Polynomials:

$\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$

is called a polynomial of degree n. $(n \in \mathbb{Z}_+)$

$(a_n \neq 0)$

eg. $10x^5 - 17x + \frac{7}{2} \rightarrow$ polynomial degree 5

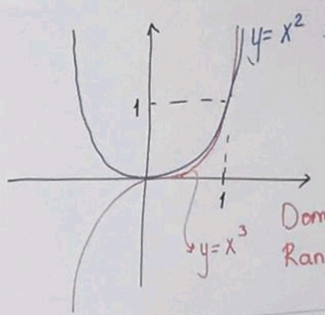
But; $\sqrt{x}$, x^{-1}, $\frac{1}{1+x}$, $x^{5/3} \rightarrow$ are not polynomials

Rational functions:

$f(x) = \frac{p(x)}{q(x)}$ > polynomials

Domain: *polynomials have domain $(-\infty, \infty)$ $(\mathbb{R})$

*rational functions have domain all x's except where $q(x) = 0$.

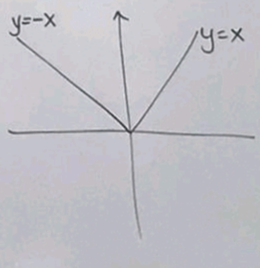


$y = x^2$: Domain: $\mathbb{R}$
Range: $[0, \infty)$

$y = x^3$ Domain: $\mathbb{R}$
Range: $(-\infty, \infty)$

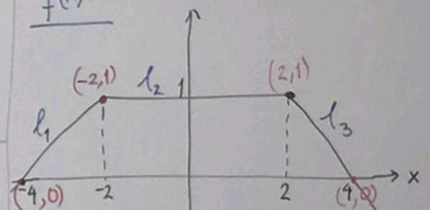
Piece-wise defined functions:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Ex. 3-2:

Find the formula of the function $f(x)$:



$$f(x) = \begin{cases} l_1, & x < -2 \\ 1, & -2 \leq x \leq 2 \\ l_3, & x > 2 \end{cases}$$

l_1: passing through $(-4,0)$ & $(-2,1)$:

$m_1 = \frac{1-0}{-2-(-4)} = \frac{1}{2}$

$y - 0 = \left(\frac{1}{2}\right)(x - (-4)) = \frac{1}{2}(x+4)$

$\Rightarrow 2y = x + 4 \Rightarrow \boxed{x - 2y = -4}$

standart line eqn. $\leftarrow ax + by = c$

$l_1: y = \frac{x+4}{2}$

$l_3: m_3 = \frac{1-0}{2-4} = \boxed{-\frac{1}{2}}$

$y - 0 = (-\frac{1}{2})(x-4)$

$y = -\frac{x-4}{2}$

$$\underbrace{f(x)}_{y} = \begin{cases} \frac{x+4}{2}, & x < -2 \\ 1, & -2 \leq x \leq 2 \\ -\frac{x-4}{2}, & x > 2 \end{cases}$$

Inverse Functions:

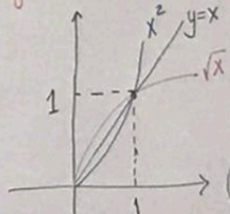
If $\underbrace{f(g(x))}_{(f \circ g)(x)} = x$ and $\underbrace{g(f(x))}_{(g \circ f)(x)} = x$

$\Rightarrow$ f & g are said to be inverses of each other (i.e $g(x) = \underbrace{f^{-1}(x)}_{\neq \frac{1}{f(x)}}$)

* The domain and range of inverse functions (in their common domains) interchange.

* Graphs of $f(x)$ & $f^{-1}(x)$ are symmetric w.r.t $y=x$

eg: Let $f(x) = x^2$, and $g(x) = \sqrt{x}$.



$x^2 = \sqrt{x} \Rightarrow x = 1$

On the common domain $[0,\infty)$ of $f(x)$ & $g(x)$, their graphs are symmetric w.r.t. $y=x$.

$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ ✓

$g(f(x)) = g(x^2) = \sqrt{x^2} = |x| = x$ ✓ (on $[0,\infty)$)

$\Rightarrow g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ on $[0,\infty)$

$\Rightarrow f(x) = x^2$ " " " $g(x) = \sqrt{x}$ " "

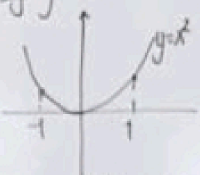
One-t-one function:

If $f(x_1)=f(x_2) \Rightarrow x_1=x_2 \Rightarrow f$ is 1-1

eg. $y=x^2$ is <u>not 1-1</u>

$f(-1)=(-1)^2=1=f(1)$

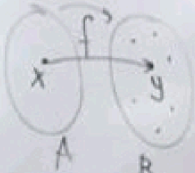
but $-1 \neq 1$



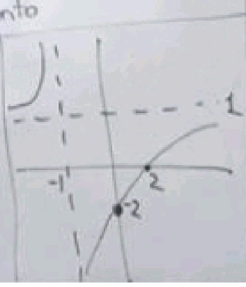
But on $[0,\infty) \Rightarrow y=f(x)=x^2$ is 1-1.

Onto: $f: A \to B$. If $\exists\ x \in A \quad \forall\ y \in B$

$\Rightarrow f(x)=y \Rightarrow f$ is onto



$f(x)=y$



Thm: A function has an inverse $\Leftrightarrow$ it is 1-1 & onto.

Ex 3.3: Find the inverse of the function $\boxed{f(x)=\frac{x-2}{x+1}}$ on

the domain: $\mathbb{R} \setminus \{-1\}$

$\left((-\infty,-1) \cup (-1,\infty)\right)$

and range: $\mathbb{R} \setminus \{1\}$

$(-\infty,1) \cup (1,\infty)$

$$y=\frac{x-2}{x+1}$$

interchange the roles of x & y:

$$x=\frac{y-2}{y+1}$$

and solve for $y \to y=f^{-1}(x)$.

$xy+x=y-2 \Rightarrow y(x-1)=-x-2$

$$\Rightarrow \boxed{y=-\frac{x+2}{x-1}} \Rightarrow \boxed{f^{-1}(x)=-\frac{x+2}{x-1}}$$

$\to y=f^{-1}(x)$

Check:

$$f(f^{-1}(x)) \overset{?}{=} x \Rightarrow f\left(-\frac{x+2}{x-1}\right)=\frac{-\frac{x+2}{x-1}-2}{-\frac{x+2}{x-1}+1}$$

$$=\frac{\frac{-x-\cancel{2}-2x+\cancel{2}}{\cancel{x-1}}}{\frac{-\cancel{x}-2+\cancel{x}-1}{\cancel{x-1}}}=\frac{-3x}{-3}=x \checkmark$$

$$f^{-1}(f(x))=f^{-1}\left(\frac{x-2}{x+1}\right)=-\frac{\frac{x-2}{x+1}+2}{\frac{x-2}{x+1}-1}=-\frac{\frac{x-\cancel{2}+2x+\cancel{2}}{\cancel{x+1}}}{\frac{\cancel{x}-2-\cancel{x}-1}{\cancel{x+1}}}$$

$$=-\frac{3x}{-3}=x \checkmark$$