ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113 - Mathematics for City Planners

2019-2020 Fall

SECOND MIDTERM EXAMINATION

10.12.2019, 17:30

SOLUTIONS

Question	Grade	Out of
1		10
2		11
3		26
4		34
5		14
6		10
Total		105

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. (10 pts.) Find the points of discontinuity (if any) of the function

$$f(x) = \begin{cases} e^x + 3 - x^2, & \text{if } x < 0 \\ 3x - 2, & \text{if } 0 \le x < 1 \\ 2 - x^2 + \ln x, & \text{if } 1 \le x \end{cases}$$

Explain your answer in detail.

at $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(e^x+3-x^2) = \underbrace{e^0}_{1}+3-0 = 4$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(3x-2) = -2$

$\Big\} \lim_{x\to 0} f(x) =$ D.N.E.

$\Rightarrow$ f is not cont. at $x=0$

at $x=1$: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3x-2) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x^2+\ln x) = 2-1+0 = 1$

$\Big\} \lim_{x\to 1} f(x) = 1 = f(1)$

$\Rightarrow$ f is cont. at $x=1$

At all other pts.; e^x+3-x^2; $3x-2$, $2-x^2+\ln x$ $(x \geq 1)$ f is cont.

$\Rightarrow$ $\boxed{\text{Only discontinuity pt. of } f \text{ is } x=0.}$

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2. (11 pts.) Find the equation of the tangent line to the curve $y^3 = x^2 - \ln(xy+1) - 8$ at the point where $x = 0$.

$x=0 \Rightarrow y^3 = 0 - \underbrace{\ln 1}_{0} - 8 = -8 \Rightarrow y=-2 \Rightarrow$ pt. $(0,-2)$

$3y^2 \cdot y' = 2x - \dfrac{1 \cdot y + x \cdot y'}{xy+1} \Rightarrow y'\left(3y^2 + \dfrac{x}{xy+1}\right) = 2x - \dfrac{y}{xy+1}$

$\Rightarrow y' = \dfrac{2x - \frac{y}{xy+1}}{3y^2 + \frac{x}{xy+1}} \Rightarrow y'\Big|_{(0,-2)} = \dfrac{0 - \frac{-2}{0+1}}{3(-2)^2 + \frac{0}{0+1}} = \dfrac{2}{12} = \boxed{\dfrac{1}{6}}$

slope of tg. line

tg. line eqn.: $y-(-2) = \dfrac{1}{6}(x-0)$

$$\boxed{y = \frac{1}{6}x - 2}$$

3. Find the <u>derivatives</u> of the following functions. Simplify your answers as much as possible.

a) (7 pts.) $f(x) = \log_{x^2}(\sqrt{x^2 - 5x}) = \dfrac{\ln (x^2-5x)^{1/2}}{\ln x^2} = \dfrac{\frac{1}{2}\ln(x^2-5x)}{2\ln x}$

$$f'(x) = \frac{1}{4}\left[\frac{\left(\frac{2x-5}{x^2-5x}\right)\cdot \ln x - \left(\frac{1}{x}\right)\ln(x^2-5x)}{(\ln x)^2}\right]$$

b) (9 pts.) $y = \left[\dfrac{1+x}{(1+x^3)e^x}\right]^{x^2}$

$$\ln y = x^2 \cdot \ln\left[\frac{1+x}{(1+x^3)e^x}\right] = x^2 \cdot \left[\ln(1+x) - \ln(1+x^3) - \ln e^x\right]$$

$$= x^2 \cdot \left[\ln(1+x) - \ln(1+x^3) - x\right]$$

$$\frac{y'}{y} = (2x)\left[\ln(1+x) - \ln(1+x^3) - x\right] + (x^2)\left[\frac{1}{1+x} - \frac{3x^2}{1+x^3} - 1\right]$$

$$\Rightarrow y' = \left[\frac{1+x}{(1+x^3)e^x}\right]^{x^2}\left[(2x)\ln(1+x) - (2x)\ln(1+x^3) - 2x^2 + \frac{x^2}{1+x} - \frac{3x^4}{1+x^3} - x^2\right]$$

c) (10 pts.) $g(x) = \dfrac{(x+2)^3(3x^2+7)\sqrt{x^2+4}}{(x^2+5x-11)(2x+1)^5}$

$$\ln g(x) = \ln (x+2)^3 + \ln(3x^2+7) + \ln(x^2+4)^{1/2} - \ln(x^2+5x-11) - \ln(2x+1)^5$$

$$= 3\ln(x+2) + \ln(3x^2+7) + \frac{1}{2}\ln(x^2+4) - \ln(x^2+5x-11) - 5\ln(2x+1)$$

$$\frac{g'(x)}{g(x)} = 3\frac{1}{x+2} + \frac{6x}{3x^2+7} + \frac{1}{2}\cdot\frac{2x}{x^2+4} - \frac{(2x+5)}{x^2+5x-11} - 5\frac{2}{2x+1}$$

$$g'(x) = g(x)\cdot\left[\frac{3}{x+2} + \frac{6x}{3x^2+7} + \frac{x}{x^2+4} - \frac{(2x+5)}{x^2+5x-11} - \frac{10}{2x+1}\right]$$

4. Find y' in the following expressions and simplify your answer as much as possible.

a) (10 pts.) $xe^y = \ln(x-y)+5$

$$1.e^y + x.e^y.y' = \frac{1-y'}{x-y} \Rightarrow y'\left[xe^y + \frac{1}{x-y}\right] = \frac{1}{x-y} - e^y$$

$$\boxed{y'} = \frac{\frac{1}{x-y} - e^y}{xe^y + \frac{1}{x-y}} = \boxed{\frac{1-xe^y+ye^y}{x^2e^y - xye^y + 1}}$$

b) (12 pts.) $y^x = x^{y^2}$ $\Rightarrow \ln y^x = \ln x^{y^2} \Rightarrow x \cdot \ln y = (y^2) \ln x$

$$\frac{d}{dx} \Rightarrow 1.\ln y + \frac{y'}{y} = (2y \cdot y')\ln x + y^2 \cdot \frac{1}{x}$$

$$y'\left[\frac{1}{y} - 2y\ln x\right] = \frac{y^2}{x} - \ln y$$

$$y' = \frac{\frac{y^2}{x} - \ln y}{\frac{1}{y} - 2y\ln x} = \frac{y^2 - x\ln y}{x} \cdot \frac{y}{1-2y^2\ln x} = \boxed{\frac{y^3 - xy\ln y}{x - 2xy^2\ln x}}$$

c) (12 pts.) $y\ln x + x\ln y = x^2y^2$

$$\left(y'\ln x + y \cdot \frac{1}{x}\right) + \left(1.\ln y + x \cdot \frac{1}{y} \cdot y'\right) = 2xy^2 + x^2 \cdot 2y \cdot y'$$

$$y'\left[\ln x + \frac{x}{y} - 2x^2y\right] = 2xy^2 - \ln y - \frac{y}{x}$$

$$\boxed{y' = \frac{2xy^2 - \ln y - \frac{y}{x}}{-2x^2y + \ln x + \frac{x}{y}}}$$

5. Given that $f(1) = 0$, $f'(1) = 3$, $f(2) = -1$, $f'(2) = -2$, $g(1) = 4$, $g'(1) = 2$, $g(2) = 1$ and $g'(2) = 5$;

a) (7 pts.) If $h(x) = \dfrac{2 + f(x)}{g(x) - x^2 + 1}$, then compute $h'(1)$.

$$h'(x) = \frac{[f'(x)][g(x) - x^2 + 1] - (g'(x) - 2x)(2 + f(x))}{(g(x) - x^2 + 1)^2}$$

$$h'(1) = \frac{[f'(1)][g(1) - 1^2 + 1] - [g'(1) - 2][2 + f(1)]}{(g(1) - 1^2 + 1)^2}$$

$$= \frac{(3)(4-0) - (2-2)(2+0)}{(4-0)^2} = \frac{12}{16} = \boxed{\frac{3}{4}}$$

b) (7 pts.) If $k(x) = f(g(2x))$, then compute $k'(1)$.

$$k'(x) = f'(g(2x)) \cdot g'(2x) \cdot 2$$

$$k'(1) = f'(g(2)) \cdot g'(2) \cdot 2$$

$$= f'(1) \cdot 5 \cdot 2 = (3)(5)(2) = \boxed{30}$$

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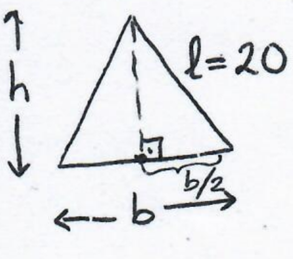
6. (10 pts.) Find the largest possible area for an isosceles triangle if the length of each of its two equal sides is 20 cm.



$$h^2 + \left(\frac{b}{2}\right)^2 = (20)^2 \Rightarrow h = \frac{\sqrt{1600 - b^2}}{2} \; ; \; A = \frac{1}{2} b \cdot h = \frac{1}{2} b \cdot \frac{\sqrt{1600 - b^2}}{2}$$

$$\frac{dA}{db} = \frac{1}{4}\left[1 \sqrt{1600 - b^2} + (b)\frac{(-\cancel{2}b)}{\cancel{2}\sqrt{1600 - b^2}}\right] = \frac{1}{4}\left(\frac{1600 - 2b^2}{\sqrt{1600 - b^2}}\right) = 0$$

$$\Rightarrow \boxed{b = 20\sqrt{2} \text{ m} \quad \text{crit. value}}$$

$$\Rightarrow h = \frac{\sqrt{1600 - 800}}{2} = \frac{20\sqrt{2}}{2} = 10\sqrt{2} \text{ m}$$

$$A = \frac{1}{2}(20\sqrt{2})(10\sqrt{2}) = \boxed{200 \text{ m}^2}$$