ÇANKAYA UNIVERSITY

Department of Mathematics

SOLUTIONS

MATH 113 - Mathematics for City Planners

2019-2020 Fall

FINAL EXAM

02.01.2020, 10:00

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 110 minutes

Question	Grade	Out of
1		12
2		12
3		12
4		10
5		18
6		16
7		9
8		21
Total		110

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 8 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Solve the following equations for **x**. Clearly explain and indicate your answers.

(6 pts.) **a)** $\log(5x+5) = \log(3x-2)+1$

$$\underbrace{\log(5x+5) - \log(3x-2)} = 1 = \log 10$$

$$\log\left(\frac{5x+5}{3x-2}\right) = \log 10 \Rightarrow \frac{5x+5}{3x-2} = 10 \Rightarrow 5x+5 = 30x-20$$

$$25 = 25x$$

$$\boxed{x = 1}$$

(6 pts.) **b)** $e^{2\ln(2x)} = 4$

$$e^{\ln(2x)^2} = 4 \Rightarrow (2x)^2 = 4 \Rightarrow 4x^2 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1$$

But for $x=-1 \Rightarrow \ln(2(-1)) = \ln(-2)$ is not defined

$\Rightarrow$ So, $\boxed{x=1 \text{ is the only solution}}$

2. Find the domains of each of the following functions:
(Indicate your answer as an interval (or union of intervals)).

(6 pts.) **a)** $f(x) = \dfrac{x}{\sqrt{x^2-9}} + \log_5(x-3)$

Dom f: $x^2-9>0$ and $x-3>0 \Rightarrow x>3$ $\Big\}$ $x>3 \Rightarrow \boxed{(3,\infty)}$

	-3		3
$x+3$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
x^2-9	$+$		$+$

(6 pts.) **b)** $g(x) = e^{\left(\frac{x^2-x-12}{x^2+5x-6}\right)} = e^{\frac{(x-4)(x+3)}{(x+6)(x-1)}}$

Dom g: $x^2+5x-6 \neq 0$

$x \neq -6$ and $x \neq 1$ $\Big\}$ Soln: $\boxed{(-\infty,-6)\cup(-6,1)\cup(1,\infty)}$

3. Evaluate the following limits. Show your work and do not use L'Hopital's rule!

(6 pts.) **a)** $\lim_{x\to\infty}\left[\frac{x^2+x+1}{(3x+2)^2}\right] = \lim_{x\to\infty}\left(\frac{x^2+x+1}{9x^2+12x+4}\right)$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(9+\frac{12}{x}+\frac{4}{x^2}\right)} = \boxed{\frac{1}{9}}$$

(6 pts.) **b)** $\lim_{x\to-\infty}(e^{-x}+2x^2+5) = e^{-(-\infty)}+2(-\infty)^2+5$

$$= \infty + \infty + 5 = \boxed{\infty}$$

(10 pts.) **4.** Let $f(x) = \begin{cases} x^2+3x+5 & \text{if } x<-2 \\ 3 & \text{if } x=-2 \\ x^3+2x^2-5x-7 & \text{if } x>-2 \end{cases}$. Find $f'(-2)$ if it exists.

$\left(\text{Hint}: f'(-2) = \lim_{x\to-2}\left[\frac{f(x)-f(-2)}{x-(-2)}\right]\right)$

$$\lim_{x\to-2^-}\left(\frac{f(x)-f(-2)}{x+2}\right) = \lim_{x\to-2^-}\left(\frac{x^2+3x+5-3}{x+2}\right) = \lim_{x\to-2^-}\frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}} = -2+1 = -1$$

$(x<-2)$

$$\lim_{x\to-2^+}\left(\frac{f(x)-f(-2)}{x+2}\right) = \lim_{x\to-2^+}\left(\frac{x^3+2x^2-5x-7-3}{x+2}\right) = \lim_{x\to-2^+}\left(\frac{\cancel{(x+2)}(x^2-5)}{\cancel{(x+2)}}\right) = (-2)^2-5 = -1$$

$(x>-2)$

$$\Rightarrow \boxed{f'(-2) = -1}$$

5. (9 pts.) **a)** Write the equation of the tangent line to the curve $x^8 + 4x^2y^2 + y^8 = 6$ at the point $(1, 1)$.

$$8x^7 + 8xy^2 + 8x^2y \cdot y' + 8y^7 \cdot y' = 0 \Rightarrow 8y'(x^2y + y^7) = -8(x^7 + xy^2)$$

$$\Rightarrow y' = -\frac{x^7 + xy^2}{y^7 + x^2y} \Rightarrow y'\Big|_{(1,1)} = -\frac{1+1}{1+1} = -1 \to \text{slope of tg. line.}$$

tg. line eqn.: $y - 1 = (-1)(x-1)$

$$y = -x + 1 + 1$$

$$\boxed{y = -x + 2}$$

(9 pts.) **b)** Let $y = \ln\left[\frac{(2x+5)(5x-2)}{(x+1)}\right]$. Find $y''(0) = ?$

$$y = \ln(2x+5) + \ln(5x-2) - \ln(x+1)$$

$$y' = \frac{2}{(2x+5)} + \frac{5}{(5x-2)} - \frac{1}{(x+1)}$$

$$y'' = \frac{(-4)}{(2x+5)^2} + \frac{(-25)}{(5x-2)^2} + \frac{1}{(x+1)^2}$$

$$y''(0) = \frac{-4}{25} + \frac{-25}{4} + 1 = \frac{-16-625}{100} + 1 = \boxed{\frac{-541}{100}}$$

6. Find y' in the following expressions. Simplify your answer as much as possible.

(8 pts.) **a)** $y=\sqrt{1+\sqrt{1+x}}$

$$y=\left(1+(1+x)^{1/2}\right)^{1/2}$$

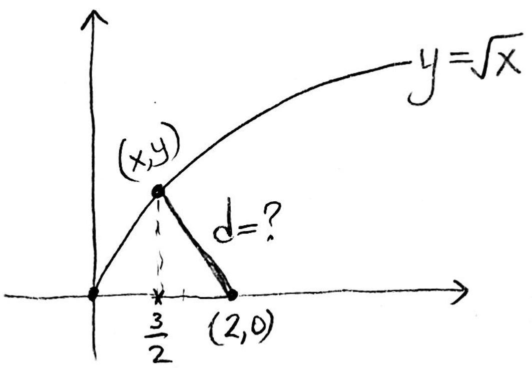
$$y'=\frac{1}{2}\left(1+(1+x)^{1/2}\right)^{-1/2}\cdot\left(\frac{1}{2}(1+x)^{-1/2}\right)=\boxed{\frac{1}{4\left(\sqrt{1+\sqrt{1+x}}\right)\left(\sqrt{1+x}\right)}}$$

(8 pts.) **b)** $y=5^{x+5\log_5 x}=5^x\cdot\underbrace{5^{\log_5 x^5}}_{x^5}=x^5\cdot 5^x$

$$y'=5x^4\cdot 5^x+x^5\cdot 5^x\ln 5$$

$$=\boxed{x^4\cdot 5^x\left[5+x\cdot\ln 5\right]}$$

(9 pts.) **7.** Find the <u>shortest distance</u> between the point $(2,0)$ and the curve $y=\sqrt{x}$.



$$d^2=(x-2)^2+(y-0)^2=(x-2)^2+y^2$$

$$=(x-2)^2+x$$

$$d^2=x^2-3x+4$$

$$\frac{d}{dx}\left(d^2(x)\right)=2x-3=0\Rightarrow\boxed{x=\frac{3}{2}}$$

$x=\frac{3}{2}$:

$$d^2=\left(\frac{3}{2}\right)^2-3\left(\frac{3}{2}\right)+4=\frac{9}{4}-\frac{9}{2}+4=\frac{9-18+16}{4}=\frac{7}{4}$$

$$d=\sqrt{\frac{7}{4}}=\boxed{\frac{\sqrt{7}}{2}}$$

8. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 3 & 4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 3 \\ 5 & 8 \\ 2 & 4 \end{bmatrix}$$

If it is possible, <u>compute the followings</u>. If it is <u>not possible</u>, explain why.

(3 pts.) **a)** $2A + B$

not possible since size $A = 2\times4$, size $B = 2\times2$ } can not add different size matrices

(3 pts.) **b)** $C^T B$

$$C^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times2} \cdot B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}_{2\times2} = \boxed{\begin{bmatrix} 5 & 6 \\ 11 & 14 \\ 15 & 18 \end{bmatrix}}$$

(3 pts.) **c)** $(CD)^T = D^T . C^T$

$$D^T . C^T = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 8 & 4 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times2} = \boxed{\begin{bmatrix} 15 & 5 \\ 31 & 8 \end{bmatrix}}$$

(3 pts.) **d)** CB

$C_{2\times3} \cdot B_{2\times2}$ = not possible since # columns of $C = 3 \neq$ # of rows of $B = 2$

(3 pts.) **e)** $CD + B$

$$\underbrace{C_{2\times3} \cdot D_{3\times2}}_{((CD)^T)^T} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 5 & 8 \\ 2 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 15 & 31 \\ 5 & 8 \end{bmatrix}}_{CD} + \underbrace{\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}}_{B} = \boxed{\begin{bmatrix} 20 & 37 \\ 6 & 10 \end{bmatrix}}$$

(3 pts.) **f)** trace B

$$= \text{trace} \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = 5 + 2 = 7$$

(3 pts.) **g)** trace C

$= \text{trace} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$: not possible since C is not a square matrix.