ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113-Mathematics for City Planners

SECOND MIDTERM EXAMINATION

13.12.2016

SOLUTIONS

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 60 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

4) Your exam will not be graded if you don't take the exam at the right place.

1. Let $f(x) = \begin{cases} \frac{x^3-2x^2-4x-16}{\sqrt{x}-2} & \text{if } x<4 \\ a & \text{if } x=4 \\ \frac{x^2+bx+c}{x-4} & \text{if } x>4 \end{cases}$. Find a, b, c if $f(x)$ is continuous at $x = 4$.

$$\lim_{\substack{x\to4^-\\(x<4)}} f(x) = \lim_{x\to4^-}\left(\frac{x^3-2x^2-4x-16}{\sqrt{x}-2}\right) = \lim_{x\to4^-}\frac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)(x^2+2x+4)}{\cancel{(\sqrt{x}-2)}}$$

$$= (2+2)(4+4+4) = \boxed{48}$$

$$\lim_{\substack{x\to4^+\\(x>4)}} f(x) = \lim_{x\to4^+}\left(\frac{x^2+bx+c}{x-4}\right) = \lim_{x\to4^+}\frac{\cancel{(x-4)}(x+4+b)}{\cancel{(x-4)}} = 4+4+b=8+b=\boxed{48}$$

for limit to exist at x=4

$$\begin{array}{l|l} \cancel{x^2}+bx+c & x-4 \\ \hline -\cancel{x^2}+4x & x+(4+b) \\ \hline \cancel{(4+b)}x+c & \\ -\cancel{(4+b)}x+4(4+b) & \\ \hline \boxed{16+4b+c=0} & \end{array}$$

$$\Rightarrow 8+b=48 \Rightarrow \boxed{b=40}$$

$$\boxed{c}=-16-4(40)=\boxed{-156}$$

For continuity at x=4: $f(4)=a=\lim_{x\to4} f(x) = 48$

$$\Rightarrow \boxed{a=48}$$

Soln.:

$$\boxed{\begin{array}{l} a=48 \\ b=40 \\ c=-156 \end{array}}$$

2. Let $f(x) = \begin{cases} x^2 - 3x + 5 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ x^3 + 2x^2 - 5x - 7 & \text{if } x > -2 \end{cases}$. Find $f'(-2)$ if it exists.

$$\boxed{f'(-2) = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)}}$$

$$x^2 - 3x + 2 = (x-2)(x-1)$$

$$\lim_{\substack{x \to -2^- \\ (x<-2)}} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^-} \left[\frac{\overbrace{(x^2 - 3x + 5) - (3)}}{(x+2)} \right] = \lim_{x \to -2^-} \frac{(x-2)(x-1)}{(x+2)} = \frac{(-4)(-3)}{0^-} = \boxed{\infty}$$

$$\lim_{\substack{x \to -2^+ \\ (x>-2)}} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^+} \frac{(x^3 + 2x^2 - 5x - 7) - 3}{(x+2)} = \lim_{x \to -2^+} \frac{\cancel{(x+2)}(x^2 - 5)}{\cancel{(x+2)}} = 4 - 5 = \boxed{-1}$$

$$\Rightarrow \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \boxed{\text{d.n.e.}} \ \left(\text{since left limit is } \infty\right)$$

$$\Rightarrow \boxed{f'(-2) \text{ d.n.e.}}$$

3. Write an equation of the tangent line drawn to the graph of $f(x) = e^{x^3-4x^2+2x+4}$ at $x = 2$.

$f(2) = e^0 = 1$

slope of tg. line $= m = f'(2) = \textcircled{-2} \leftarrow$

$$f'(x) = e^{x^3-4x^2+2x+4} \cdot (3x^2-8x+2)$$

$$f'(2) = e^{2^3-4(2)^2+2(2)+4} \cdot (3.4-8.2+2) = e^0.(12-16+2) = 1.(-2) = \textcircled{-2}$$

$\Rightarrow$ tg. line: (at (2,1)) $y-1 = (-2)(x-2) \Rightarrow y = -2x+4+1$

$$\searrow \boxed{y = -2x+5}$$

4. (a) Find $f'(x)$ if $f(x) = e^{x^3} + \ln(x^2 + 2x + 3)$.

$$f'(x) = e^{x^3} \cdot 3x^2 + \frac{2x+2}{x^2+2x+3} = \boxed{3x^2 \cdot e^{x^3} + \frac{2(x+1)}{x^2+2x+3}}$$

(b) Find $f'(x)$ if $f(x) = (x-1)^2 \ln x$.

$$f'(x) = \boxed{2(x-1) \cdot \ln x + (x-1)^2 \cdot \frac{1}{x}}$$

(c) Find $f'(x)$ if $f(x) = \frac{\ln x}{e^x}$.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(e^x) - e^x \cdot \ln x}{(e^x)^2} = \frac{\cancel{e^x}\left[1 - x\ln x\right]}{x \cdot \cancel{e^{2x}}} = \boxed{\frac{1 - x\ln x}{x \cdot e^x}}$$

(d) Find $f'(x)$ if $f(x) = \frac{x^2+1}{xe^x}$.

$$f'(x) = \frac{(2x)(xe^x) - (x^2+1)(e^x + xe^x)}{(x \cdot e^x)^2}$$

$$= \frac{2x^2 \cdot e^x - x^2e^x - x^3e^x - e^x - xe^x}{x^2 \cdot e^{2x}}$$

$$= \frac{\cancel{e^x}\left[x^2 - x^3 - x\right]}{x^2 \cdot \cancel{e^{2x}}} = \frac{\cancel{x}(x - x^2 - 1)}{\cancel{x^2} \cdot e^x} = \boxed{\frac{x - x^2 - 1}{x \cdot e^x}}$$

5. (a) Find y' if $x^3 + y^3 = 5$.

y' means $y = y(x) \Rightarrow 3x^2 + 3y^2 \cdot y' = 0$

$$y' = -\frac{\cancel{3}x^2}{\cancel{3}y^2} = \boxed{-\frac{x^2}{y^2}}$$

(b) Find y' if $3x^2y^2 + 2xy^2 + y^2 + x^3 = 0$.

$$3[2xy^2 + x^2 \cdot 2yy'] + 2[y^2 + x \cdot 2y \cdot y'] + 2y \cdot y' + 3x^2 = 0$$

$$y'[6x^2y + 4xy + 2y] = -3x^2 - 2y^2 - 6xy^2$$

$$\boxed{y' = -\frac{3x^2 + 2y^2 + 6xy^2}{4xy + 2y + 6x^2y}}$$

(c) Find y' if $ye^{xy} + \ln(xy^2) + xy = 2$.

$$y'.e^{xy} + y.e^{xy}.\left[y + xy'\right] + \frac{y^2 + x.2y.y'}{xy^2} + (y + xy') = 0$$

$$y'\left[e^{xy} + xye^{xy} + \frac{2}{y} + x\right] = -y^2.e^{xy} - \frac{1}{x} - y$$

$$\boxed{y' = -\frac{y^2.e^{xy} + \frac{1}{x} + y}{e^{xy} + xye^{xy} + \frac{2}{y} + 1}}$$

(d) Find y' if $x^3y + \overbrace{\ln y^x}^{x.\ln y} + e^{x^2y} = 3$.

$$3x^2.y + x^3y' + \left[\ln y + x.\frac{y'}{y}\right] + e^{x^2y}.\left[2xy + x^2.y'\right] = 0$$

$$y'\left[x^3 + \frac{x}{y} + x^2.e^{x^2y}\right] = -2xye^{x^2y} - 3x^2y - \ln y$$

$$\boxed{y' = -\frac{2xye^{x^2y} + 3x^2y + \ln y}{x^3 + \frac{x}{y} + x^2.e^{x^2y}}}$$