ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113-Mathematics for City Planners

FINAL EXAMINATION

03.01.2017

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 60 minutes

Question	Grade	Out of
1		16
2		16
3		15
4		15
5		18
6		20
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

4) Your exam will not be graded if you don't take the exam at the right place.

1. Find the domain of the given function.

(a) $f(x) = \sqrt{x^2 - 4x - 5}$.

$$\text{Dom} f = \{x \mid \underbrace{x^2 - 4x - 5}_{(x-5)(x+1)} \geq 0\} = \boxed{(-\infty, -1] \cup [5, \infty)}$$

x		-1		5	
$x+1$	$-$	0	$+$		$+$
$x-5$	$-$		$-$	0	$+$
	$+$		$-$		$+$

(b) $f(x) = \sqrt[7]{3x - 5}$. $\Rightarrow \boxed{\text{Dom} f = \mathbb{R}}$

(c) $f(x) = e^{\frac{x^2 - x - 2}{x^2 + 3x - 4}}$.

$$\left.\begin{array}{l} x^2 + 3x - 4 = (x+4)(x-1) \\ x^2 - x - 2 = (x-2)(x+1) \end{array}\right\} \frac{x^2 - x - 2}{x^2 + 3x - 4} = \frac{(x-2)(x+1)}{(x+4)(x-1)}$$

not defined when $x = 1$ or $x = -4$

So $\text{Dom} f = \mathbb{R} \setminus \{1, -4\}$

$= \boxed{(-\infty, -4) \cup (-4, 1) \cup (1, \infty)}$

(d) $f(x) = \ln \frac{(x+1)(x-2)}{(x+4)(x-5)} = \ln(x+1) + \ln(x-2) - \ln(x+4) - \ln(x-5)$

$x+1 > 0$, $x > -1$ $\quad$ $x-2 > 0$, $x > 2$ $\quad$ $x+4 > 0$, $x > -4$ $\quad$ $x-5 > 0$, $x > 5$

$-4 \quad -1 \quad 2 \quad 5$

So if $x > 5$ all the ln terms will be defined

$\Rightarrow \text{Dom} f(x) : (5, \infty)$

2. Do not use *L'Hôpital's* rule.

(a) Evaluate $\lim_{x\to -1} \frac{x^3+x^2+3x+3}{x^3+1}$ $= \lim_{x\to -1} \left[\frac{x^2(x+1)+3(x+1)}{(x+1)(x^2-x+1)}\right]$

$$= \lim_{x\to -1} \left[\frac{\cancel{(x+1)}(x^2+3)}{\cancel{(x+1)}(x^2-x+1)}\right] = \frac{(-1)^2+3}{(-1)^2-(-1)+1} = \boxed{\frac{4}{3}}$$

(b) Evaluate $\lim_{x\to 4} \frac{x^3-2x^2-5x-12}{\sqrt{x}-2}$ $= \lim_{x\to 4} \frac{(x-4)(x^2+2x+3)}{\sqrt{x}-2}$

$$x^3-2x^2-5x-12 \,\big|\, x-4$$
$$-x^3 \pm 4x^2 \qquad x^2+2x+3$$
$$2x^2-5x$$
$$-2x^2 \pm 8x$$
$$3x-12$$
$$3x-12$$
$$0 \quad 0$$

$$= \lim_{x\to 4} \frac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)(x^2+2x+3)}{\cancel{(\sqrt{x}-2)}}$$

$$= (2+2)(16+8+3) = 4 \times 27 = \boxed{108}$$

(c) Evaluate $\lim_{x\to\infty} \frac{5x^4-2x^3+x}{3x^3-7x-x^4+2}$.

$$= \lim_{x\to\infty} \frac{\cancel{x^4}\left(5-\frac{2}{x}+\frac{1}{x^3}\right)}{\cancel{x^4}\left(\frac{3}{x}-\frac{7}{x^3}-1+\frac{2}{x^4}\right)} = \frac{5-0+0}{0-0-1+0} = \frac{5}{-1} = -5$$

(d) Let $f(x) = \begin{cases} \frac{x^2+2x}{x^2+3x+2} & \text{if } x < -2 \\ a & \text{if } x = -2 \\ \frac{x^2+bx+c}{x^2-2x-8} & \text{if } x > -2 \end{cases}$. Find a, b, c if $f(x)$ is continuous at $x = -2$.

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2} \left(\frac{x^2+2x}{x^2+3x+2}\right) = \frac{0}{0} = \lim_{x\to -2} \left(\frac{x\cancel{(x+2)}}{(x+1)\cancel{(x+2)}}\right) = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

3. (a) Let $f(x) = \begin{cases} x^2 - 7x + 20 & \text{if } x < 2 \\ 10 & \text{if } x = 2 \\ -6\sqrt{2x} + 22 & \text{if } x > 2 \end{cases}$. Find $f'(2)$ if it exists.

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{(x^2-7x+20)-10}{x-2} = \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x-5)}{\cancel{(x-2)}} = -3$$

left der. at x=2

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(-6\sqrt{2x}+22)-10}{x-2} = \lim_{x\to 2^+} \frac{-6(\sqrt{2x}-2)}{x-2}$$

right der. at x=

$$= \lim_{x\to 2^+} \frac{-6(\sqrt{2x}-2)(\sqrt{2x}+2)}{(x-2)(\sqrt{2x}+2)} = \lim_{x\to 2^+} \frac{-6\overbrace{(2x-4)}^{=2\cancel{(x-2)}}}{\cancel{(x-2)}(\sqrt{2x}+2)} = \frac{-12}{2+2} = -3$$

$\Rightarrow$ since right & left derivs. are the same,

$f'(2)=3$

(b) Find $f'(x)$ if $f(x) = e^{x^3} + \ln(2x^2+3)$.

$$\boxed{f'(x) = e^{x^3}(3x^2) + \frac{4x}{2x^2+3}} = \boxed{3x^2 \cdot e^{x^3} + \frac{4x}{2x^2+3}}$$

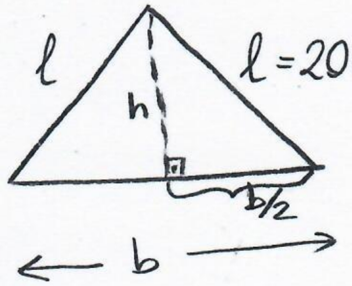
(c) Find y' if $x^3y + e^{xy^2} = 5$.

$$3x^2y + x^3y' + e^{xy^2}(1.y^2 + x.2y\,y') = 0$$

$$y'\left[x^3 + 2xy\,e^{xy^2}\right] = -3x^2y - y^2e^{xy^2}$$

$$\boxed{y' = -\frac{3x^2y + y^2e^{xy^2}}{x^3 + 2xy\,e^{xy^2}}}$$

4. Find the largest possible area for an isosceles triangle if the length of each of its two equal sides is 20 m.



$$A = \frac{1}{2} b.h = \frac{1}{2} b.\frac{\sqrt{1600-b^2}}{2}$$

$$h^2 + \left(\frac{b}{2}\right)^2 = (20)^2$$

$$h^2 = 400 - \frac{b^2}{4} \Rightarrow h = \sqrt{\frac{1600-b^2}{4}} = \frac{\sqrt{1600-b^2}}{2}$$

$$\frac{dA}{db} = \frac{1}{4}\left[1.\sqrt{1600-b^2} + b.\frac{(-\cancel{2}b)}{\cancel{2}\sqrt{1600-b^2}}\right]$$

$$= \frac{1}{4}\left[\frac{1600-b^2-b^2}{\sqrt{1600-b^2}}\right] = 0 \Rightarrow 2b^2 = 1600 \Rightarrow b = 2\sqrt{2} \times 10 = 20\sqrt{2}$$

$$\boxed{b = 20\sqrt{2}\text{ m.}}$$

crit. value

$$\Rightarrow h = \frac{\sqrt{1600-800}}{2} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}\text{ m}$$

$$A = \frac{1}{2}(20\sqrt{2})(10\sqrt{2}) = \boxed{200\text{ m}^2}$$

5. Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$

a) Find A^2.

$$A^2 = A.A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 27 & 4 & 12 \\ 0 & 11 & 6 \\ 10 & 8 & 25 \end{bmatrix}$$

b) Find AB.

$$A.B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 10 & -1 \\ 8 & -9 & -1 \\ 11 & -1 & 12 \end{bmatrix}$$

c) Find BA.

$$B.A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 21 \\ -9 & 3 & 2 \\ 10 & 5 & -1 \end{bmatrix}$$

d) Find $A^T B$.

$$A^T.B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & -1 \\ 5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 2 \\ 11 & -11 & 1 \\ 14 & -3 & 14 \end{bmatrix}$$

e) Find $B^T A^T$.

$$B^T.A^T = (A \cdot B)^T = \begin{bmatrix} 6 & 8 & 11 \\ 10 & -9 & -1 \\ -1 & -1 & 12 \end{bmatrix}$$

f) Find $2A + B$.

$$2\begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 2 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 12 \\ 5 & 3 & -4 \\ 8 & 1 & 5 \end{bmatrix}$$

6. Find the trace of the following matrices, it possible.

a) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} = 1+(-1)+(-4) = -4$

b) $\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}^T = \text{trace} \begin{bmatrix} 3 & 0 & 3 \\ -2 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = (3)+(-2)+(2) = 3$

c) $\left(\underbrace{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}}_{\text{not possible}} \right)^T \longrightarrow X$

d) $\underset{2\times3}{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 3 & -2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}} = \begin{bmatrix} 12 & -1 \\ 3 & 5 \end{bmatrix}$

$tr \begin{bmatrix} 12 & -1 \\ 3 & 5 \end{bmatrix} = 12+5 = 17$