$\boxed{\text{M1:}}$ Tuesday; Nov. 16, 2021
17:30
L111

(math 111. cankaya.edu.tr
exercises: W1 – $\boxed{\text{W5}}$
(One-side limits, continuity)

4.17) $$\lim_{x\to\infty}\left(\frac{8e^x}{4+5e^x}\right)=\frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\left\{\frac{\cancel{e^x}(8)}{\cancel{e^x}\left(\frac{4}{e^x}+5\right)}\right\}=\frac{8}{0+5}=\boxed{\frac{8}{5}}$$

$\frac{4}{e^x} \to 0$

$\frac{4}{\infty}=0$

4.18) $$\lim_{x\to\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-10x+1}\right)=\infty-\infty$$

$$=\lim_{x\to\infty}\left\{\frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-10x+1}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-10x+1}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-10x+1}\right)}\right\}$$

$$=\lim_{x\to\infty}\left\{\frac{\overbrace{(x^2+4x)-(x^2-10x+1)}^{(14x-1)}}{\sqrt{x^2\left(1+\frac{4}{x}\right)}+\sqrt{x^2\left(1-\frac{10}{x}+\frac{1}{x^2}\right)}}\right\}$$

$$x\sqrt{1+\frac{4}{x}}+x\sqrt{1-\frac{10}{x}+\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\left\{\frac{\cancel{x}\left(14-\frac{1}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{10}{x}+\frac{1}{x^2}}\right)}\right\}$$

$\frac{1}{x}\to 0$, $\frac{4}{x}\to 0$, $\frac{10}{x}\to 0$, $\frac{1}{x^2}\to 0$

$$=\frac{14-0}{\sqrt{1+0}+\sqrt{1-0+0}}=\frac{14}{2}=\boxed{7}$$

Instead if we have:

$$\lim_{x\to-\infty}\left(\sqrt{x^2+4x}-\sqrt{x^2-10x+1}\right)=\infty-\infty$$

$=$ multiply & divide by the conjugate to get $\Big\}$ $=\lim_{x\to-\infty}\left\{\dfrac{\cancel{x}\left(14-\frac{1}{x}\right)}{-\cancel{x}\left(\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{10}{x}+\frac{1}{x^2}}\right)}\right\}$

using $\sqrt{x^2}=(-x)$ as $x\to-\infty$

$$=\frac{14-0}{-\left(\sqrt{1+0}+\sqrt{1-0+0}\right)}=\frac{14}{-2}=-7$$

4.21) $\lim_{x\to\infty}\dfrac{1}{2x-\sqrt{4x^2-5x+6}}=\lim_{x\to\infty}\left\{\cancel{\dfrac{1}{2x\left(1-\sqrt{1-\frac{5}{4x^2}+\frac{6}{4x^2}}\right)}}\right\}$?

$\dfrac{1}{\infty\cdot 0}$?

$$=\lim_{x\to\infty}\left(\frac{\left(2x+\sqrt{4x^2-5x+6}\right)}{4x^2-\left(4x^2-5x+6\right)}\right)=\lim_{x\to\infty}\left(\frac{\cancel{x}\left(2+\sqrt{4-\frac{5}{x}+\frac{6}{x^2}}\right)}{\cancel{x}\left(5-\frac{6}{x}\right)}\right)$$

$$=\frac{2+\sqrt{4-0+0}}{5-0}=\frac{2+2}{5}=\frac{4}{5}$$

4.22) $\lim_{x\to\infty}\left(\sqrt{2x^2-1}-\sqrt{x^2+1}\right)=\infty-\infty$

$$=\lim_{x\to\infty}\frac{\overbrace{(2x^2-1)-(x^2+1)}^{x^2-2}}{\sqrt{2x^2-1}+\sqrt{x^2+1}}\underset{\sqrt{x^2}=x}{=}\lim_{x\to\infty}\frac{x^{\cancel{2}}\left(1-\frac{2}{x^2}\right)}{\cancel{x}\left(\sqrt{2-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)}=\frac{\infty(1-0)}{\sqrt{2-0}+\sqrt{1+0}}=\frac{\infty}{\sqrt{2}+1}=\infty$$

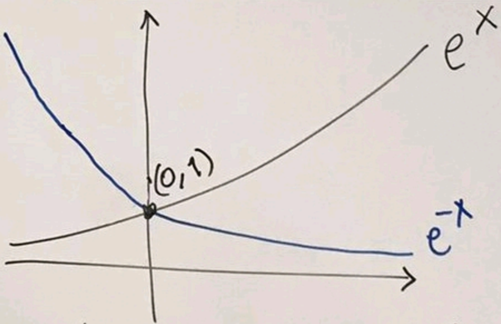
$$\sqrt{x^2}=|x|=\begin{cases} x, & x>0\\ -x, & x\leq 0\end{cases}$$

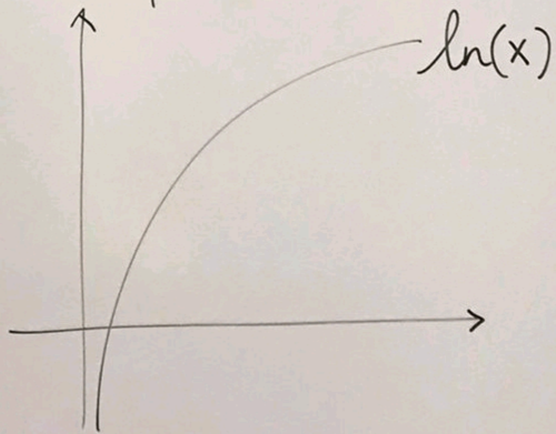
$$\lim_{x\to\infty}(\quad):\ \sqrt{x^2}=x$$

$$\lim_{x\to-\infty}(\quad):\ \sqrt{x^2}=\boxed{(-x)}$$

$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty$: indeterminate forms

do not use L'Hôpital's rule





Ex. 5.5: Evaluate the limit if it exists.

DO NOT USE L'hôpital's rule!

$$\lim_{x\to 8^+}\left(\frac{x^2-10x+16}{\sqrt{x-8}}\right) = \left(\frac{0}{0}\right)$$

Note that $\lim\limits_{x\to 8}\ldots$ results in $\lim\limits_{x\to 8^+}\ldots$

Since $\sqrt{x-8}$ is defined for $x>8$

$$=\lim_{x\to 8^+}\left(\frac{(x-8)(x-2)}{\sqrt{x-8}}\right) = \lim_{x\to 8^+}\left(\frac{\cancel{(\sqrt{x-8})}^2(x-2)}{\cancel{\sqrt{x-8}}}\right)$$

$$= 0\cdot 6 = \boxed{0}$$

Continuity:

We say that f is continuous at "a" if

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

That is;

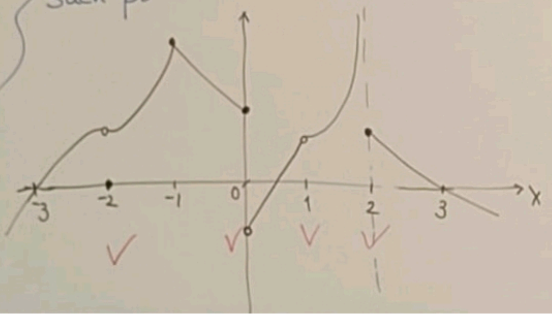
* f is defined at "a"
* $\lim\limits_{x\to a} f(x)$ exists
* $\lim\limits_{x\to a} f(x) = f(a)$

Discontinuity pts. are:

$\{-2, 0, 1, 2\}$

Ex. 5.6:

Determine the pts. where $f(x)$ is discontinuous and explain the reason of discontinuity at each such pt.



$\underline{x=-2}$: $f(-2)$ is not equal to the limit value of $f(x)$ at $x=-2$

$\underline{x=0}$: limit d.n.e. at $x=0$ (since right and left limits are not equal)

$\underline{x=1}$: f is not defined at $x=1$

$\underline{x=2}$: $\lim_{x\to 2} f(x)$ = d.n.e. since $\lim_{x\to 2^-} f(x) = \infty$

$\underline{\text{Ex.5.7:}}$

$$f(x)=\begin{cases} 2x^2+a & \text{if } x<2 \\ b & \text{if } x=2 \\ 3x-2 & \text{if } x>2 \end{cases}$$

Find "a" & "b" such that f is continuous at $x=2$ (and hence, everywhere).

* $f(2)=b$
* $\lim_{x\to 2} f(x)$:

i) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x^2+a) = \boxed{8+a}$

ii) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x-2) = \boxed{4}$

$\boxed{\lim_{x\to 2} f(x)} = 8+a = \boxed{4} \Rightarrow \boxed{a=-4}$

* $b = f(2) = \lim_{x\to 2} f(x) = 4$

$\Rightarrow \boxed{b=4}$

Ex.5-8:

$$f(x)=\begin{cases} \log_{10}\left(\frac{x}{2}+b\right), & x<8 \\ x\left(\sqrt{x-8}+\frac{1}{4}\right), & x\geq 8 \end{cases}$$

Find b if $f(x)$ is continuous at $x=8$

$* \; f(8)=8\left(\sqrt{0}+\frac{1}{4}\right)=\boxed{2}=\lim_{x\to 2^+} f(x)$

$* \; \lim_{x\to 8^-} f(x)=\lim_{x\to 8^-}\log\left(\frac{x}{2}+b\right)=\boxed{\log(4+b)}$

$\Rightarrow$ For continuity at $x=8$: $\log_{10}(4+b)=2$

$\Rightarrow 4+b=10^2=100 \Rightarrow \boxed{b=96}$

$D_f=\left\{x \mid \frac{x}{2}+96>0\right\} \Rightarrow x>-2\times 96=-192$

$=(-192,\infty)$

5-23) $f(x)=\begin{cases} x^2-c^2, & \text{if } x\leq 1 \\ (x-c)^2, & \text{if } x>1 \end{cases}$

$f(1)=1^2-c^2=\boxed{1-c^2}=\lim\limits_{x\to 1^-} f(x)$

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-c)^2=\boxed{(1-c)^2}$

$\Rightarrow \cancel{1}-c^2=(1-c)^2=\cancel{1}-2c+c^2$

$\Rightarrow 2c=2c^2 \Rightarrow c^2-c=0$

$c(c-1)=0$

$\Rightarrow \boxed{c=0 \text{ or } c=1}$

5-14) $f(x)=\dfrac{|x-a|}{x-a}=\begin{cases} \dfrac{x-a}{x-a}=1, & x>a \\ \dfrac{-(x-a)}{x-a}=-1, & x<a \end{cases}$

$\left.\begin{array}{l} \lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^-}(-1)=-1 \\ \qquad\qquad\qquad\qquad \neq \\ \lim\limits_{x\to a^+} f(x)=\lim\limits_{x\to a^+}(1)=1 \end{array}\right\}$ lim d.n.e. at $x=a$

$\Rightarrow f$ is discont. at $x=a$

5-16) $f(x)=\dfrac{1}{e^{2x}-e^{3x}}$

f is discont. at those x's for which f is undefined $\Rightarrow e^{2x}=e^{3x} \Rightarrow \underset{\neq 0}{e^{2x}}\underbrace{(1-e^{x})}_{=0}=0$

$\Rightarrow e^{x}=1 \Rightarrow x=0$

} f is discont. at $x=0$

5.17) $f(x) = \dfrac{x-5}{x^2-25} = \dfrac{x-5}{(x-5)(x+5)} \underset{x\neq 5}{=} \dfrac{1}{x+5}$

Discont. pts.: $x = \mp 5$

$x=-5$: f undefined

$x=5$: " "

5.18) $f(x) = \dfrac{1}{1-|x|}$

f is undefined when $|x| = 1 \Rightarrow x = \mp 1$

f is discont. at $x = \mp 1$

eg. $f(x) = \dfrac{1}{a-|2x|}$

f is discont.: $|2x| = \mp a \Rightarrow x = \mp \dfrac{a}{2}$

$|x| = \begin{cases} x, & x>0 \\ -x, & x\leq 0 \end{cases}$

5.11) $\lim\limits_{x\to 0^+} \dfrac{2x^2+3x|x|}{x|x|} \underset{x>0}{=} \lim\limits_{x\to 0^+} \dfrac{2x^2+3x^2}{x^2} = 5$

5.12) $\lim\limits_{x\to 0^-} \dfrac{2x^2+3x|x|}{x|x|} \underset{x<0}{=} \lim\limits_{x\to 0^-} \dfrac{2x^2-3x^2}{-x^2} = \lim\limits_{x\to 0^-} \left(\dfrac{-x^2}{-x^2}\right) = 1$

$\Rightarrow f(x) = \dfrac{2x^2+3x|x|}{x|x|}$ is discontinuous at $x=0$

since left & right limits at $x=0$ are not the same $\Rightarrow \lim\limits_{x\to 0} f(x) =$ d.n.e.