

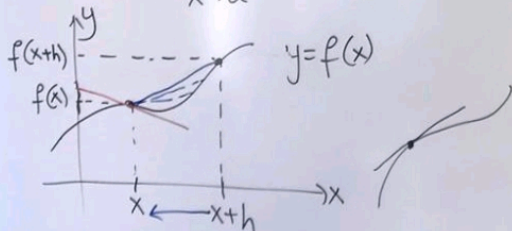
Derivatives:

Defn.: The derivative of the function $f(x)$ is the function $f'(x)$ defined by;

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{defn. of the derivative}$$

or equivalently;

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$



We can think of the derivative as;

- * the rate of change of the function f , or
- * slope of the curve (i.e. tangent line) of $y = f(x)$

We will use; y' , $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$ to denote derivatives;

and; $f'(a)$, $\left. \frac{dy}{dx} \right|_{x=a}$ to denote their values at $x=a$

* Note that; derivative is a function, its value at a point is a number.

Higher-order derivatives:

Second-order derivative: y'' , $f''(x)$, $\frac{d^2y}{dx^2}$

higher-order ders.: $y'''(x)$, $y^{(4)}(x) = y^{(IV)}(x)$, \dots , $y^{(n)}(x)$

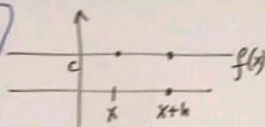
Differentiation Formulas: using the defn. of derivative

we obtain:

* derivative of a constant is zero:

$$\boxed{f(x)=C} \Rightarrow \boxed{f'(x)=0}$$

$$f(x)=C, f(x+h)=C$$



$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{C - C}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \boxed{0}$$

* derivative of $\boxed{f(x)=x}$ is $\boxed{f'(x)=1}$

$$f(x)=x, f(x+h)=x+h$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h) - (x)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = \boxed{1}$$

* derivative of $f(x)=x^n$ is $f'(x)=nx^{n-1}$:

$n \in \mathbb{Z}$:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \dots + h^n) - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + n(n-1)x^{n-2}h^2 + \dots + h^{n-1}}{h} \quad \text{(n terms involving h's)} \\ &= \lim_{h \rightarrow 0} \frac{h(n x^{n-1} + n(n-1)x^{n-2}h + \dots + h^{n-2})}{h} = \boxed{nx^{n-1}} \end{aligned}$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

$$\vdots$$

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \dots + nxh^{n-1} + h^n$$

(n+1)-terms

$$f(x) = x^n$$

$$\Rightarrow f'(x) = nx^{n-1}$$

More generally; if $n \in \mathbb{R} \Rightarrow f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
 power rule

$$* f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

* If f is a function and c is a constant \Rightarrow

$$\checkmark \frac{d}{dx}(cf(x)) = cf'(x)$$

* If f & g are functions; \Rightarrow

$$\checkmark (f \mp g)' = f' \mp g'$$

Ex. 6.1: $f(x) = 7x^3 - 18x$

$$f'(x) = 7(3x^2) - 18(1) = 21x^2 - 18$$

$$f''(x) = 42x$$

$$f'''(x) = 42 \Rightarrow f^{(4)}(x) = 0 \Rightarrow f^{(n)}(x) = 0, n \geq 4$$

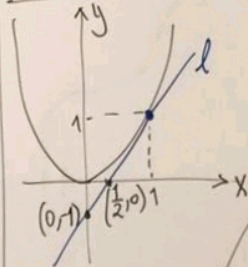
Ex. 6.2: $f(x) = \frac{7x^3 - 18x}{x} \Rightarrow f'(x) = ?$

$$f(x) = \frac{7x^3}{x} - \frac{18x}{x} = 7x^2 - 18 \Rightarrow f'(x) = 14x$$

Ex. 6.3: Find the eqn. of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$.

Soln.:

slope of $t = f'(1) = m$



$$f'(x) = 2x$$

$$f'(1) = 2$$

$$y - 1 = 2(x - 1)$$

$$y = 2x - 2 + 1 = 2x - 1$$

eqn. of tg. line: $y = 2x - 1$

Differentiation Rules:

Product rule:

If f & g are differentiable at $x \Rightarrow$

$(f \cdot g)$ is also differentiable at x &

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f) \cdot g + f \cdot \frac{d}{dx}(g)$$

$$\Rightarrow \boxed{(f \cdot g)' = f' \cdot g + g' \cdot f}$$

Ex. 64: $f(x) = (x^4 + 14x)(7x^3 + 17)$

$$f'(x) = (4x^3 + 14)(7x^3 + 17) + (x^4 + 14x)(21x^2)$$

Quotient rule: $g(x) \neq 0$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{(g)^2}$$

Ex. 67: $f(x) = \frac{2x+3}{5x^2+7} \Rightarrow f'(x) = \frac{(2)(5x^2+7) - (10x)(2x+3)}{(5x^2+7)^2} = \frac{10x^2+14-20x^2-30x}{(5x^2+7)^2}$

$$\Rightarrow f'(x) = \frac{-10x^2 - 30x + 14}{(5x^2+7)^2} = \boxed{\frac{10x^2 + 30x - 14}{(5x^2+7)^2}}$$

Reciprocal rule: $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$

Ex. 65: Find the der. of $f(x) = \frac{1}{x^n}$

$$\Rightarrow f'(x) = -\frac{n x^{n-1}}{x^{2n}} = -n x^{n-1-2n} = -n x^{-1-n} = \boxed{\frac{-n}{x^{n+1}}}$$

Ex. 66: $f(x) = \frac{1}{8x^2+12x+1} \Rightarrow f'(x) = ?$

$$f'(x) = \frac{-(16x+12)}{(8x^2+12x+1)^2} = \boxed{\frac{-4(4x+3)}{(8x^2+12x+1)^2}}$$

Ders. of Exponentials & Logarithms:

$$\boxed{\frac{d}{dx}(e^x) = e^x} \Rightarrow \boxed{\frac{d}{dx}(a^x) = a^x} \quad (a > 0, a \neq 1)$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \Rightarrow \boxed{\frac{d}{dx}(\log_b x) = \frac{1}{dx} \left(\frac{\ln x}{\ln b} \right)} \quad (b > 0, b \neq 1)$$

$$= \boxed{\frac{1}{\ln b} \cdot \frac{1}{x}}$$

$$\left(\log_b x = \frac{\ln x}{\ln b} \right)$$

change-of-basis formula

Exercises:

6.12) $f(x) = \frac{x^{3/2} + x^{-1/2}}{\sqrt{x} + \frac{1}{\sqrt{x}}} \Rightarrow f'(x) = ?$

$$f(x) = \frac{x^{3/2} + \frac{1}{x^{1/2}}}{x^{1/2} + \frac{1}{x^{1/2}}} = \frac{\frac{x^2 + 1}{x^{1/2}}}{\frac{x + 1}{x^{1/2}}}$$

$$f(x) = \frac{x^2 + 1}{x + 1}$$

$$f'(x) = \frac{(2x)(x+1) - (1)(x^2+1)}{(x+1)^2}$$

$$= \frac{2x^2 + 2x - x^2 - 1}{(x+1)^2}$$

$$f'(x) = \frac{x^2 + 2x - 1}{(x+1)^2}$$

6.14) $f(x) = x^2 \cdot \ln(x^3) = 3[x^2 \cdot \ln(x)]$

$$f'(x) = 3 \left[(2x)(\ln x) + (x^2) \left(\frac{1}{x} \right) \right]$$

$$= 3(2x)(\ln x) + 3x$$

$$f'(x) = (2x) \ln(x^3) + 3x$$

$$[f(x) \cdot g(x) \cdot h(x)]' = (f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h')(x)$$

6.19) $f(x) = x^4 \cdot e^x \cdot \ln x$
 $\Rightarrow f'(x) = ?$

$$f'(x) = (4x^3)(e^x)(\ln x) + (x^4)(e^x) \left(\frac{1}{x} \right) + (x^4 \cdot e^x) \cdot \ln x$$

$$= 4x^3 \cdot e^x \cdot \ln x + x^4 \cdot e^x \cdot \ln x + x^3 \cdot e^x$$

$$6-22) f(x) = \frac{1}{\ln(4x)} \Rightarrow f'(x) = \frac{-(\ln(4x))'}{(\ln(4x))^2} = \frac{-\frac{1}{x}}{[\ln(4x)]^2}$$

reciprocal rule

$$[\ln(4x)]' = [\ln 4 + \ln x]' = 0 + \frac{1}{x} = \frac{1}{x}$$

Eqn. of tg. line to $f(x)$ at x_0 :

$$6-44) f(x) = 2x^2 - 8x + 4, x_0 = 2$$

$$m = f'(2) = 4(2) - 8 = 0$$

$$f'(x) = 4x - 8$$

$$(x_0, y_0) = (2, f(2)) = (2, -4)$$

$$f(2) = 2(2)^2 - 8(2) + 4 = 8 - 16 + 4 = -4$$

$$y - y_0 = m(x - x_0) = 0 \quad y - (-4) = 0$$

$$\boxed{y = -4}$$

$$6-47) f(x) = \frac{1}{1+x^2}, x_0 = 0 \Rightarrow f(0) = 1 \Rightarrow$$

tg. line at $(0, 1)$:

$$f'(x) = -\frac{2x}{(1+x^2)^2} \Rightarrow f'(0) = -\frac{0}{1^2} = 0$$

$$\left. \begin{array}{l} \text{tg. line: } y - 1 = (0)(x - 0) \\ \Rightarrow \boxed{y = 1} \end{array} \right\}$$

$$6-51) f'(a) = ? \quad f(x) = \frac{\ln x}{x^4}, a = 1$$

$$f'(x) = \frac{(\frac{1}{x})(x^4) - (4x^3)(\ln x)}{(x^4)^2}$$

$$= \frac{x^3 - 4x^3 \ln x}{(x^4)^2 = x^8}$$

$$f'(1) = \frac{1^3 - 4(1^3)(\ln 1)}{1^8} = \frac{1 - 0}{1} = \boxed{1}$$

$$6-52) f(x) = (1+2x)(e^x), a = 0 \Rightarrow f'(0) = ?$$

$$f'(x) = (2)(e^x) + (1+2x)(e^x) = e^x[2+1+2x] = (3+2x)(e^x)$$

$$f'(0) = (3+2(0))(e^0) = \boxed{3}$$