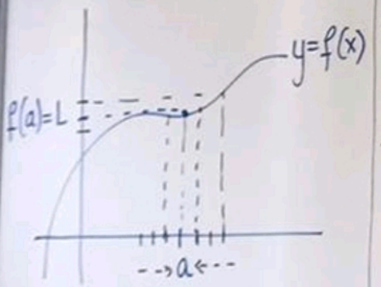
Limits:



$$\boxed{\lim_{x\to a} f(x) = L}$$

* If x approaches "a" from valves larger than "a" we write $x \to a^+$ $(x>a)$
* " " " " " " smaller " " " " $x \to a^-$ $(x<a)$

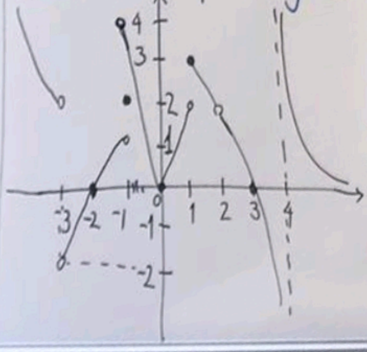
⇒ * If f approaches L_1 as $\boxed{x \to a^+}$ we write $\boxed{\lim_{\substack{x\to a^+ \\ (x>a)}} f(x) = L_1}$ and call this the right-limit of $f(x)$.

* " " " L_2 as $\boxed{x \to a^-}$ " " $\boxed{\lim_{\substack{x\to a^- \\ (x<a)}} f(x) = L_2}$ " " " " left-limit of $f(x)$.

* $\boxed{\lim_{x\to a} f(x) = L} \iff \boxed{\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)}$

$$b \neq 0: \begin{cases} \dfrac{b}{0} = \begin{cases} +\infty & (b>0) \\ -\infty & (b<0) \end{cases} \\ \dfrac{0}{b} = 0 \end{cases}$$

Ex: Find the following limits and values using the graph of $f(x)$:



$f(-3) = $ undefined	$\lim_{\substack{x\to -3^- \\ (x<-3)}} f(x) = 2$	$\lim_{\substack{x\to -3^+ \\ (x>-3)}} f(x) = -2$	$\Rightarrow \lim_{x\to -3} f(x) = $ d.n.e.
$f(-2) = 0$	$\lim_{x\to -2^-} f(x) = 0$	$\lim_{x\to -2^+} f(x) = 0$	$\Rightarrow \lim_{x\to -2} f(x) = 0$

$f(-1)=2$	$\lim\limits_{x\to -1^-} f(x) = 1$ $(x<-1)$	$\lim\limits_{x\to -1^+} f(x) = 4$ $(x>-1)$	$\Rightarrow \lim\limits_{x\to -1} f(x) =$ d.n.e.
$f(0)=0$	$\lim\limits_{x\to 0^-} f(x) = 0$	$\lim\limits_{x\to 0^+} f(x) = 0$	$\lim\limits_{x\to 0} f(x) = 0$
$f(1)=3$	$\lim\limits_{x\to 1^-} f(x) = 2$	$\lim\limits_{x\to 1^+} f(x) = 3$	$\Rightarrow \lim\limits_{x\to 1} f(x) =$ d.n.e.
$f(2)=$ undefined	$\lim\limits_{x\to 2^-} f(x) = 2$	$\lim\limits_{x\to 2^+} f(x) = 2$	$\Rightarrow \lim\limits_{x\to 2} f(x) = 2$
$f(3)=0$	$\lim\limits_{x\to 3^-} f(x) = 0$	$\lim\limits_{x\to 3^+} f(x) = 0$	$\lim\limits_{x\to 3} f(x) = 0$
$f(4)=$ undefined	$\lim\limits_{x\to 4^-} f(x) = -\infty$	$\lim\limits_{x\to 4^+} f(x) = \infty$ $(+\infty)$	$\lim\limits_{x\to 4} f(x) =$ d.n.e.

Properties of Limits:

1.) If $f(x) = c$ (constant) $\Rightarrow \lim\limits_{x\to a} f(x) = c$

2.) $\lim\limits_{x\to a} x^n = a^n$ for any positive integer n

(exp. func., logar. func.)

3.) If f is a polynomial function $\Rightarrow$

$$\lim_{x\to a} f(x) = f(a)$$

Suppose $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$

4.) $\lim\limits_{x\to a} \left(f(x) \mp g(x)\right) = L \mp M$

5.) $\lim\limits_{x\to a} \left(f(x)\cdot g(x)\right) = L \cdot M$

6.) $\lim\limits_{x\to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$ $(M \neq 0)$

9.) $\lim\limits_{x\to a} f(g(x)) = \boxed{f(M)}$

7-) $\lim_{x\to a} \left(c \cdot f(x)\right) = c \cdot L$ $\left(c \to \text{constant}\right)$

8.) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} = (L)^{1/n}$

Examples:

1-) $\lim_{x\to -3}\left(\frac{x-3}{x+5}\right) = \frac{-3-3}{-3+5} = \frac{-6}{2} = -3$

2-) $\lim_{x\to 0}\left(\frac{x}{x^3-4x+3}\right) = \frac{0}{0-0+3} = \frac{0}{3} = \boxed{0}$

$a^2-b^2=(a-b)(a+b)$

$(a+b)^2=a^2+2ab+b^2$

$(a-b)=a^2-2ab+b^2$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

3.) $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1}(x+1) = 2$

$\downarrow$ $\left(\frac{0}{0}\right)$ $(x\neq 1)$

4.) $\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right) = \lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = 3$

$\downarrow$ $\left(\frac{0}{0}\right)$ $(x\neq 1)$

5.) $\lim_{x\to 2}\left(\frac{x^2-5x+6}{x^2-3x+2}\right) = \lim_{x\to 2}\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-1)} = \frac{-1}{1} = -1$

$\downarrow$ $\left(\frac{0}{0}\right)$ $(x\neq 2)$

6.) $\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right) = \lim_{x\to 1}\frac{\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)} = \frac{1}{2}$

$\downarrow$ $\left(\frac{0}{0}\right)$ $x\neq 1$

$x-1=(\sqrt{x}-1)(\sqrt{x}+1)$

$(\sqrt{x})^2-1$

7-) $\lim_{x\to e} \ln(4x^2) = \ln(4e^2) = \ln 4 + \underbrace{\ln e^2}_{2} = \boxed{2+\ln 4}$

8-) $\lim_{x\to 4} \log_2\left[\left(\frac{x}{x-2}\right)^2\right] = \log_2\left(\frac{4}{2}\right)^2 = 2\log_2 2 = 2$

9.) $\lim_{x\to 2}\left(e^x + 2^x + 4x\right) = e^2 + 2^2 + 4(2)$

$= \boxed{12+e^2}$

Further examples:

1.-) $\lim\limits_{x\to 1}\left(\dfrac{x^2-5x+6}{x^2-3x+2}\right) = \dfrac{2}{0}$: $\boxed{\text{d.n.e.}}$

2.-) $\lim\limits_{x\to 0}\dfrac{x}{|x|}$:

$x<0 \Downarrow |x|=-x \Rightarrow \lim\limits_{x\to 0^-}\dfrac{x}{|x|} = \lim\limits_{x\to 0^-}\left[\dfrac{x}{(-x)}\right] = \lim\limits_{x\to 0^-}(-1) = \boxed{-1}$

$\neq$

$x>0 \Downarrow |x|=x \rightarrow \lim\limits_{\substack{x\to 0^+\\(x>0)}}\dfrac{x}{|x|} = \lim\limits_{x\to 0^+}\left[\dfrac{x}{x}\right] = \lim\limits_{x\to 0^+}(1) = \boxed{1}$

$\Longrightarrow$ left-limit $\neq$ right-limit $\Longrightarrow \lim\limits_{x\to 0}\dfrac{x}{|x|}$: $\boxed{\text{d.n.e}}$

3.-) Let $f(x) = \begin{cases} 2-x^2 & \text{if } x>1 \\ -2+3x & \text{if } 0\le x\le 1 \\ 4-x^2 & \text{if } x<0 \end{cases}$

$3x=2$

a) $\lim\limits_{x\to 0} f(x) =$ d.n.e.

b.) $\lim\limits_{x\to 1} f(x) = 1$

a.-) $\lim\limits_{\substack{x\to 0^-\\(x<0)}} f(x) = \lim\limits_{x\to 0^-}(4-x^2) = 4$ | $\lim\limits_{\substack{x\to 0^+\\(x>0)}} f(x) = \lim\limits_{x\to 0^+}(-2+3x) = -2$

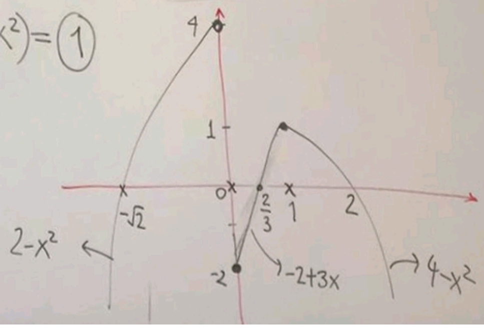
$\neq$

$\Longrightarrow \lim\limits_{x\to 0} f(x)$: d.n.e.

b.-) $\lim\limits_{\substack{x\to 1^-\\(x<1)}} f(x) = \lim\limits_{x\to 1^-}(-2+3x) = 1$

$\lim\limits_{\substack{x\to 1^+\\(x>1)}} f(x) = \lim\limits_{x\to 1^+}(2-x^2) = 1$

$= \Longrightarrow \boxed{\lim\limits_{x\to 1} f(x) = 1}$



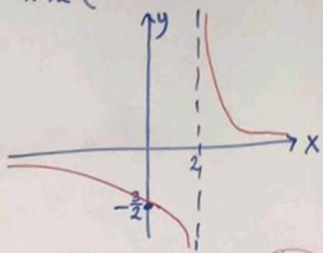
4-) $$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \underset{\left(\frac{0}{0}\right)}{\overset{\downarrow}{=}} \lim_{x\to 0}\left[\frac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x(\sqrt{x+9}+3)}\right] = \lim_{x\to 0}\frac{((x+9)-9)}{x(\sqrt{x+9}+3)} = \lim_{\substack{x\to 0 \\ x\neq 0}} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+9}+3)} = \boxed{\frac{1}{6}}$$

5-) $$\lim_{x\to 1}\left(\frac{\sqrt{15+x}-4}{x-1}\right) \underset{\left(\frac{0}{0}\right)}{\overset{\downarrow}{=}} \lim_{x\to 1}\left[\frac{(\sqrt{15+x}-4)(\sqrt{15+x}+4)}{(x-1)(\sqrt{15+x}+4)}\right] = \lim_{\substack{x\to 1 \\ x\neq 1}} \frac{\overbrace{(15+x)-16}^{\cancel{x-1}}}{\cancel{(x-1)}(\sqrt{15+x}+4)} = \frac{1}{8}$$

6-) $$\lim_{x\to 5}\left[\frac{(\sqrt{x-1}-2)}{x^2-6x+5}\right] \underset{\left(\frac{0}{0}\right)}{\overset{\downarrow}{=}} \lim_{x\to 5}\left[\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{(x-5)(x-1)(\sqrt{x-1}+2)}\right] = \lim_{\substack{x\to 5 \\ x\neq 5}}\left[\frac{\overbrace{((x-1)-4)}^{\cancel{(x-5)}}}{\cancel{(x-5)}(x-1)(\sqrt{x-1}+2)}\right] = \frac{1}{(4)(4)} = \frac{1}{16}$$

7) $$\lim_{x\to 2}\frac{(\sqrt{3x+10}-4)}{(x-2)} \underset{\left(\frac{0}{0}\right)}{\overset{\downarrow}{=}} \lim_{x\to 2}\frac{(\sqrt{3x+10}-4)(\sqrt{3x+10}+4)}{(x-2)(\sqrt{3x+10}+4)} = \cdots = \frac{3}{8}$$

8) $$\lim_{x\to 2}\left(\frac{3}{x-2}\right) = \text{d.n.e.}$$



$$\lim_{x\to 2^-}\left(\frac{3}{x-2}\right) = \frac{3}{0^-} = -\infty$$
$\left(\begin{array}{l} x<2 \\ x-2<0 \end{array}\right.$

$$\lim_{x\to -\infty}\left(\frac{3}{x-2}\right) = \frac{3}{-\infty} = 0^-$$

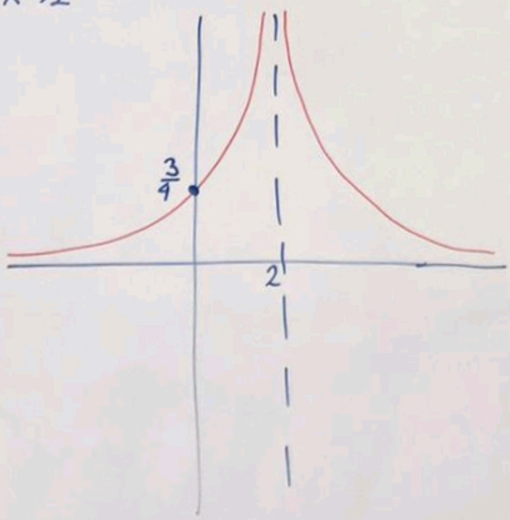
$$\lim_{x\to 2^+}\left(\frac{3}{x-2}\right) = \frac{3}{0^+} = +\infty = \infty$$
$x>2$

$$\lim_{x\to \infty}\left(\frac{3}{x-2}\right) = \frac{3}{\infty} = 0^+$$

$$\lim_{x\to 2} \frac{3}{(x-2)^2} = \text{d.n.e.}$$

$$\lim_{x\to 2^-} \frac{3}{(x-2)^2} = \frac{3}{(0^-)^2} = \infty$$

$$\lim_{x\to 2^+} \frac{3}{(x-2)^2} = \frac{3}{(0^+)^2} = \infty$$



$$\lim_{x\to -\infty} \frac{3}{(x-2)^2} = \frac{3}{(-\infty)^2} = \frac{3}{\infty} = 0^+$$

$$\lim_{x\to \infty} \frac{3}{(x-2)^2} = \frac{3}{\infty^2} = \frac{3}{\infty} = 0^+$$