Product of two matrices:

$$A_{m\times n} \cdot B_{n\times p} = [A \cdot B]_{m\times p}$$

# of rows: m	$\boxed{\text{# of rows: } n}$
$\boxed{\text{# of columns: } n}$	# of columns: p

But if $p \neq m \Rightarrow$

$B_{n\times p} \cdot A_{m\times n}$: is undefined

So; in general $A \cdot B \neq B \cdot A$ for matrices; moreover when $A \cdot B$ is defined $\Rightarrow B \cdot A$ may be undefined.

Product of 2 matrices is possible $\Leftrightarrow$ # of columns of the first matrix is the same as the # of rows of the second matrix.

ex: $\overbrace{\begin{bmatrix}3 & -2 & 1\\ 0 & -2 & 1\end{bmatrix}}^{A}_{2\times \boxed{3}} \cdot \overbrace{\begin{bmatrix}3 & -2\\ 0 & -2\\ 3 & 1\end{bmatrix}}^{B}_{\boxed{3}\times 2} = \underbrace{\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}_{2\times 2}}_{C} = \begin{bmatrix}12 & -1\\ 3 & 5\end{bmatrix}_{2\times 2}$ $(A \cdot B = C)$

$C_{11} = (3)(3) + (-2)(0) + (1)(3) = 12$ (sum of the product of entries of the 1^{st} row of matrix A & 1^{st} column of matrix B)

$C_{12} = (3)(-2) + (-2)(-2) + (1)(1) = -1$

$C_{21} = (0)(3) + (-2)(0) + (1)(3) = 3$

$C_{22} = (0)(-2) + (-2)(-2) + (1)(1) = 5$

$$B.A=\begin{bmatrix}3 & -2\\ 0 & -2\\ 3 & 1\end{bmatrix}\cdot\begin{bmatrix}3 & -2 & 1\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}_{3\times3}=\begin{bmatrix}9 & -2 & 1\\ 0 & 4 & -2\\ 9 & -8 & 4\end{bmatrix}$$

$3\times2 = 2\times3$

$C_{11}=(3)(3)+(-2)(0)=9$ $\quad C_{12}=(3)(-2)+(-2)(-2)=-2$ $\quad C_{13}=(3)(1)+(-2)(1)=1$

$C_{21}=(0)(3)+(-2)(0)=0$ $\quad C_{22}=(0)(-2)+(-2)(-2)=4$ $\quad C_{23}=(0)(1)+(-2)(1)=-2$

$C_{31}=(3)(3)+(1)(0)=9$ $\quad C_{32}=(3)(-2)+(1)(-2)=-8$ $\quad C_{33}=(3)(1)+(1)(1)=4$

* Transpose of product of two matrices is the product of the transposes of the two matrices in the reverse order! $\Rightarrow (A\cdot B)^T = B^T\cdot A^T$

$$[A_{m\times n}\cdot B_{n\times p}]^T=[C_{m\times p}]^T=[\tilde{C}]_{p\times m}$$

$$[B]_{n\times p}\Rightarrow[B^T]_{p\times n} \qquad [A]_{m\times n}\Rightarrow[A^T]_{n\times m}$$

$$\underbrace{[B^T]_{p\times n}\cdot[A^T]_{n\times m}}_{[D]_{p\times m}}$$

eg. $C=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\end{bmatrix}_{2\times3}$ $\quad D=\begin{bmatrix}-1 & 3\\ 5 & 8\\ 2 & 4\end{bmatrix}_{3\times2}$

$$C.D=\begin{bmatrix}15 & 31\\ 5 & 8\end{bmatrix}$$

i) $[C.D]^T=\begin{bmatrix}15 & 5\\ 31 & 8\end{bmatrix}$

ii) $D^T.C^T=?$

$D^T=\begin{bmatrix}-1 & 5 & 2\\ 3 & 8 & 4\end{bmatrix}_{2\times3}$ $\quad C^T=\begin{bmatrix}1 & 0\\ 2 & 1\\ 3 & 0\end{bmatrix}_{3\times2}$

$$D^T.C^T=\begin{bmatrix}15 & 5\\ 31 & 8\end{bmatrix}$$

Fall 2020-Math 113 Final Exam:

1) Find the domain:

a) $f(x)=\sqrt{x^2-4x-5}$

$$\text{Dom } f=\{x \mid \underbrace{x^2-4x-5}_{(x-5)(x+1)}\geq 0\}=(-\infty,-1]\cup[5,\infty)$$

x	-1	5	
x+1	−	+	+
x−5	−	−	+
(x+1)(x−5)	+	−	+

b) $f(x)=\ln\left[\dfrac{(x+1)(x-2)}{(x+4)(x-5)}\right]$

$\text{Dom } f: \{x \mid x>5\} = \boxed{(5,\infty)}$

$$=\underbrace{\ln(x+1)}_{>0}+\underbrace{\ln(x-2)}_{>0}-\underbrace{\ln(x+4)}_{>0}-\underbrace{\ln(x-5)}_{>0}$$

$x>-1$ $\quad x>2$ $\quad x>-4$ $\quad x>5$

-4 -1 2 5

c) $g(x)=e^{\left(\frac{x^2-x-2}{x^2+3x-4}\right)}$

$$\text{Dom } g=\{x \mid \frac{x^2-x-2}{\underbrace{x^2+3x-4}_{(x+4)(x-1)}} \text{ is defined}\}$$

$=\{x \mid x\neq 1 \text{ \& } x\neq -4\}$

-4 1

$=\boxed{(-\infty,-4)\cup(-4,1)\cup(1,\infty)}$

2) Evaluate the following limits (Do not use L'Hôpital's rule)

a) $\lim\limits_{x\to -1}\left(\dfrac{x^3+x^2+3x+3}{x^3+1}\right)=\left(\dfrac{0}{0}\right)$

$$\begin{array}{l|l} x^3+x^2+3x+3 & x+1 \\ \hline -x^3\mp x^2 & x^2+3 \\ 0\ 0+3x+3 & \\ \quad -3x\mp 3 & \\ \quad 0\ 0 & \end{array}$$

$x^3+1=(x+1)(x^2+x+1)$

$$\lim_{x\to -1}\frac{\cancel{(x+1)}(x^2+3)}{\cancel{(x+1)}(x^2\mp x+1)}=\frac{4}{1-1+1}=4$$

b) $\lim\limits_{x\to 4}\left(\dfrac{x^3-2x^2-5x-12}{\sqrt{x}-2}\right)=\dfrac{0}{0}$

$$\begin{array}{l|l} x^3-2x^2-5x-12 & x-4 \\ \hline -x^3\mp 4x^2 & x^2+2x+3 \\ \quad 2x^2-5x & \\ \quad -2x^2\mp 8x & \\ \qquad 3x-12 & \\ \qquad -3x\mp 12 & \\ \qquad 0\ 0 & \end{array}$$

$$=\lim_{x\to 4}\frac{\overbrace{(x-4)}^{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}(x^2+2x+3)}{\cancel{(\sqrt{x}-2)}}$$

$=(\sqrt{4}+2)(4^2+2(4)+3)=(4)(27)=108$

c) $\lim\limits_{x\to\infty}\left(\dfrac{5x^4-2x^3+x}{3x^3-7x-x^4+2}\right)=\dfrac{\infty}{\infty}$

$$=\lim_{x\to\infty}\left[\frac{\cancel{x^4}\left(5-\frac{2}{x}+\frac{1}{x^3}\right)}{\cancel{x^4}\left(\frac{3}{x}-\frac{7}{x^3}-1+\frac{2}{x^4}\right)}\right]=\frac{5-0+0}{0-0-1+0}=\frac{5}{-1}=\boxed{-5}$$

(d) $f(x)=\begin{cases} \dfrac{x^2+2x}{x^2+3x+2}, & \text{if } x<-2 \\ a, & \text{if } x=-2 \\ \dfrac{x^2+bx+c}{x^2-2x-8}, & \text{if } x>-2 \end{cases}$

Find the values of the constants a, b, c if $f(x)$ is cont. at $x=-2$.

$$\lim_{\substack{x\to-2^- \\ (x<-2)}} f(x)=\lim_{x\to-2^-}\left(\frac{x^2+2x}{x^2+3x+2}\right)$$

$$=\lim_{x\to-2^-}\left[\frac{x\cancel{(x+2)}}{\cancel{(x+2)}(x+1)}\right]=\frac{-2}{-2+1}=\boxed{2}$$

For cont.:

$$a=f(-2)=\lim_{x\to-2}f(x)=\begin{cases}\lim_{x\to-2^-}f(x)=2\\ \lim_{x\to-2^+}f(x)=?\end{cases}$$

$$\Rightarrow \boxed{a=2}$$

$$\lim_{x\to-2^+}f(x)=\lim_{x\to-2}\left(\frac{x^2+bx+c}{x^2-2x-8}\right)=\lim_{x\to-2}\frac{x^2+bx+c}{(x-4)(x+2)}=2$$

|| must be (for cont.)

2

$$\begin{array}{l|l} x^2+bx+c & x+2 \\ \hline -x^2 \mp 2x & x+(b-2) \\ \hline (b-2)x+c & \\ -(b-2)x \mp 2(b-2) & \\ \hline \end{array}$$

$\boxed{c-2b+4=0}$

$\boxed{2b-c=4}$

$$=\lim_{x\to-2}\frac{\cancel{(x+2)}(x+(b-2))}{\cancel{(x+2)}(x-4)}=2$$

$$=\frac{-2+(b-2)}{-2-4}=\frac{b-4}{-6}=2\Rightarrow b-4=-12\Rightarrow\boxed{b=-8}$$

$$c=2b-4=2(-8)-4=\boxed{-20}$$

$$\boxed{b=-8,\ c=-20}$$

(3) $f(x)=\begin{cases}x^2-7x+20, & \text{if } x<2\\ 10, & \text{if } x=2\\ -6\sqrt{2x}+22, & \text{if } x>2\end{cases}$

Find $f'(2)$ if it exists?

$$\left(f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)$$

$x^2-7x+10=(x-5)(x-2)$

$$\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to2^-}\frac{\overbrace{(x^2-7x+20)-10}}{x-2}$$

$$=\lim_{x\to2^-}\frac{(x-5)\cancel{(x-2)}}{\cancel{(x-2)}}=2-5=\boxed{-3}\ \left(\begin{array}{c}\text{left deriv.}\\ \text{at } x=2\end{array}\right)$$

$$\lim_{x\to 2^+}\left(\frac{f(x)-f(2)}{x-2}\right)=\lim_{\substack{x\to 2^+\\(x>2)}}\frac{(-6\sqrt{2x}+22)-(10)}{x-2}$$

$$=\lim_{x\to 2^+}\frac{-6(\sqrt{2x}-2)}{(x-2)}\cdot\frac{(\sqrt{2x}+2)}{(\sqrt{2x}+2)}=\lim_{x\to 2^+}\left[\frac{-6\overbrace{(2x-4)}^{2(\cancel{x-2})}}{\cancel{(x-2)}(\sqrt{2x}+2)}\right]$$

$$=\lim_{x\to 2^+}\left[\frac{(-6)(2)}{\sqrt{2x}+2}\right]=\frac{-12}{2+2}=\boxed{-3}\Rightarrow$$ right der. of f at $x=2$

$$\Rightarrow\ \boxed{f'(2)=-3}$$

④ ⓐ Find $f'(x)$ if $f(x)=e^{x^3}+\ln(2x^2+3)$

$$f'(x)=e^{x^3}\cdot(3x^2)+\frac{1}{2x^2+3}\cdot(4x)$$

ⓑ Find y' if $x^3y+e^{xy^2}=5$

$$\left[(3x^2)(y)+(x^3)(y')\right]+e^{xy^2}\cdot\left[(1)(y^2)+(x)(2y\cdot y')\right]=0$$

$$y'(x^3+2xye^{xy^2})=-3x^2y-y^2e^{xy^2}$$

$$y'=-\frac{3x^2y+y^2e^{xy^2}}{x^3+2xye^{xy^2}}$$

⑤ $A=\begin{bmatrix}1&3&5\\2&2&-1\\4&-1&2\end{bmatrix}$, $B=\begin{bmatrix}3&-2&2\\1&-1&-2\\0&3&1\end{bmatrix}$

ⓐ $$A^2=A\cdot A=\begin{bmatrix}1&3&5\\2&2&-1\\4&-1&2\end{bmatrix}\begin{bmatrix}1&3&5\\2&2&-1\\4&-1&2\end{bmatrix}=\begin{bmatrix}27&4&12\\2&11&6\\10&8&25\end{bmatrix}$$

ⓑ $$A\cdot B=\begin{bmatrix}6&10&1\\8&-7&-1\\11&-1&12\end{bmatrix}$$

(c) $B^T \cdot A^T = ?$

$$\underbrace{B^T \cdot A^T}_{\parallel} \quad (A \cdot B)^T = \begin{bmatrix} 6 & 8 & 11 \\ 10 & -7 & -1 \\ 1 & -1 & 12 \end{bmatrix}$$

(d) $$3A-2B = \begin{bmatrix} 3 & 9 & 15 \\ 6 & 6 & -3 \\ 12 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 6 & -4 & 4 \\ 2 & -2 & -4 \\ 0 & 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 13 & 11 \\ 4 & 8 & 1 \\ 12 & -9 & 4 \end{bmatrix}$$

6) Find the trace of the following matrices, if possible!

(a) $$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \Rightarrow \text{trace}(A) = 1+(-1)+(-4) = -4$$

(b) $$\text{trace}\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}^T = (3)+(-2)+(2) = 3$$

(c) $$\Bigg(\underbrace{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}}_{\text{undefined}}\Bigg)^T \Rightarrow \text{trace}(\quad)^T:$$

not possible

(d) $$\underset{2\times 3}{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}} \cdot \underset{3\times 2}{\begin{bmatrix} 3 & -2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

trace

Bonus Q: $m = y'(2) = (2x-2)\big|_2 = 2$

Eqn. of tg line to: $y = x^2 - 2x + 5$ at $x_0 = 2 \rightarrow y_0 = 5$

$$\boxed{y-5 = 2(x-2)}$$