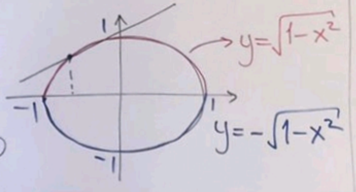
Implicit Differentiation:

An eqn. involving x & y may define y as a function of x. This is called an implicit function. For example; the following eqns. define y implicitly;

* $x^2+y^2=1$
* $ye^y+2x-\ln y=0$
* $3xy+x^2y^3+x=5$
* $e^x+e^y=\sqrt{x+2y}$



$y=y(x)$

$$\underbrace{\frac{d}{dx}(y^n)=ny^{n-1}\cdot y'(x)}_{\text{chain rule}}$$

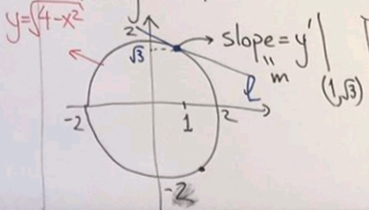
The following eqns. define y explicitly:

* $y=x^3-5x^2$, * $y=\ln(x^2-e^x)$, * $y=x^3+\sqrt{x}+xe^x$, . . .

The derivative of y can be found without solving for y. This is called implicit differentiation. The main idea is:

* differentiate each term w.r.t. "x"
* solve for y'

Ex.7.1: Find the slope of the tangent line to the curve $x^2+y^2=4$ at the point $(1,\sqrt{3})$



$$2x+2y\cdot y'=0 \Rightarrow \cancel{2}y\cdot y'=-\cancel{2}x$$

$$y'\Big|_{(1,\sqrt{3})}=-\frac{x}{y}\Big|_{(1,\sqrt{3})}=\boxed{-\frac{1}{\sqrt{3}}}$$

tg. line ℓ: $y-\sqrt{3}=\left(-\frac{1}{\sqrt{3}}\right)(x-1)$

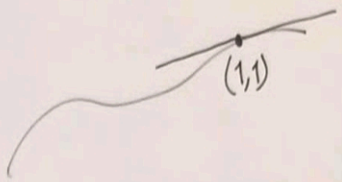
$$y=-\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}+\sqrt{3}$$

$$\boxed{y=\frac{-1}{\sqrt{3}}x+\frac{4}{\sqrt{3}}}$$

Ex. 7.2: Find the slope of the tangent line to the curve

$$\boxed{x^8+4x^2y^2+y^8=6}$$

at the pt. $(1,1)$.



$\underbrace{\text{slope of tg. line}}_{m} = y'\Big|_{(1,1)} = ?\ \Rightarrow \frac{d}{dx}(x^8)+4\frac{d}{dx}(x^2y^2)+\frac{d}{dx}(y^8)=\frac{d}{dx}(6)$

$$\Rightarrow 8x^7+4\left[(2x)(y^2)+(x^2)(2y\cdot y')\right]+(8y^7\cdot y')=0$$

$$y'\left[8x^2y+8y^7\right]=-8x^7-8xy^2 \Rightarrow y'=-\frac{\cancel{8}(x^7+xy^2)}{\cancel{8}(y^7+x^2y)}$$

$$\Rightarrow y'\Big|_{(1,1)}=-\frac{x^7+xy^2}{y^7+x^2y}\Bigg|_{(1,1)}=-\frac{1+1}{1+1}=\boxed{-1}$$

tg. line eqn.: $y-1=(-1)(x-1) \Rightarrow y=-x+1+1$

$$\Rightarrow \boxed{y=-x+2}$$

Ex. 7.3: Find y' at $(0,0)$ where

$$[(1+x+2y)][e^y] + 3[x][e^y] = 1 + x^2 + y^2$$

$$\left[\underbrace{(1+2y')(e^y) + (1+x+2y)(e^y\cdot y')}_{\text{product rule}}\right] + 3\left[\underbrace{(1)(e^y) + (x)\cdot(e^y\cdot y')}_{\text{product rule}}\right] = 0 + 2x + 2y\cdot y'$$

$$y'\left[2e^y + (1+x+2y)e^y + 3xe^y - 2y\right] = 2x - 3e^y - e^y$$

$$y' = \frac{2x - 4e^y}{e^y[3+4x+2y] - 2y}$$

$$y'\Big|_{(0,0)} = \frac{0 - 4\overset{1}{e^0}}{\underset{1}{e^0}[3+0+0] - 2(0)} = \boxed{\frac{-4}{3}}$$

tg. line:

$$y - 0 = \left(-\frac{4}{3}\right)(x - 0)$$

$$\boxed{3y + 4x = 0}$$

Find y' using implicit differentiation:

$\hookrightarrow y = y(x)$

7.4 $x = y + y^{2/3} \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(y) + \frac{d}{dx}\left(y^{2/3}\right)$

$\Rightarrow 1 = y' + \frac{2}{3} y^{-1/3} \cdot y' \Rightarrow y'\left[1 + \frac{2}{3\sqrt[3]{y}}\right] = 1$

$$\Rightarrow y' = \frac{1}{\frac{3\sqrt[3]{y}+2}{3\sqrt[3]{y}}} = \boxed{\frac{3\sqrt[3]{y}}{3\sqrt[3]{y}+2}}$$

7.7 $e^{xy} = x + 2y$

$e^{xy} \cdot [(1)(y) + (x)(y')] = 1 + 2y'$

$$y'\left[x \cdot e^{xy} - 2\right] = 1 - ye^{xy} \Rightarrow \boxed{y' = \frac{1 - ye^{xy}}{xe^{xy} - 2}}$$

7.9 $y^2 \cdot \ln y = x^3 \cdot e^x$

Using (product rule + chain rule) within "implicit diff." :

$$\left[(2y \cdot y')(\ln y) + (y^{\cancel{2}}) \cdot \left(\frac{1}{\cancel{y}} \cdot y'\right)\right] = (3x^2)(e^x) + (x^3)(e^x)$$

$$y'\underbrace{\left[(2y)(\ln y) + (y)\right]}_{y[2\ln y + 1]} = x^2 e^x (3 + x)$$

$$\boxed{y' = \frac{x^2 e^x (3 + x)}{y(2\ln y + 1)}}$$

(7.12) $x^{1/3}+y^{1/5}=y$

$\frac{1}{3}x^{-2/3}+\frac{1}{5}y^{-4/5}\cdot y'=y'$

$\frac{1}{3x^{2/3}}=y'\left[1-\frac{1}{5y^{4/5}}\right]$

$y'=\dfrac{\frac{1}{3x^{2/3}}}{\left(\frac{5y^{4/5}-1}{5y^{4/5}}\right)}=\boxed{\dfrac{5y^{4/5}}{3x^{2/3}(5y^{4/5}-1)}}$

Find y' at the indicated pt:

(or slope of the tg. line at the given pt.)

(7.16) $\underbrace{\sqrt{11+y^2}}_{(11+y^2)^{1/2}}-12xy+2y^2+4x=0$ at $(1,5)$

$\frac{1}{\cancel{2}}(11+y^2)^{-1/2}\cdot(\cancel{2}y\cdot y')-12[(1)(y)+(x)(y')]+2(2yy')+4=0$

$\frac{y\cdot y'}{\sqrt{11+y^2}}-12y-12xy'+4yy'+4=0$

$y'\left[\frac{y}{\sqrt{11+y^2}}-12x+4y\right]=-4+12y$

$y'\Big|_{(1,5)}=\dfrac{-4+12y}{\left(\frac{y}{\sqrt{11+y^2}}-12x+4y\right)}\Bigg|_{(1,5)}=\dfrac{-4+60}{\frac{5}{6}-12+20}=\boxed{\dfrac{336}{53}}$

7.17 $xe^x - ye^y + xy - 1 = 0$ at $(1,1)$

$$(1\cdot e^x + x\cdot e^x) - (y'\cdot e^y + (y)(e^y\cdot y')) + (1\cdot y + x\cdot y') = 0$$

$$y'[x - e^y - ye^y] = -e^x - xe^x - y$$

$$y'\Big|_{(1,1)} = -\frac{e^x + xe^x + y}{x - e^y - ye^y}\Big|_{(1,1)} = -\frac{e+e+1}{1-e-e} = \boxed{\frac{2e+1}{2e-1}}$$

7.18 $\ln(xy) + xy^2 - \ln(3x) - 6y = 0$, $(2,3)$

$$\frac{1}{xy}[1\cdot y + x\cdot y'] + [1\cdot y^2 + x\cdot(2yy')] - \frac{1}{\cancel{3}x}(\cancel{3}) - 6y' = 0$$

$$y'\left[\frac{\cancel{x}}{\cancel{x}y} + 2xy - 6\right] = \frac{1}{x} - \frac{y}{xy} - y^2$$

$$y'\left[\frac{1}{y} + 2xy - 6\right] = -y^2 \Rightarrow y'\Big|_{(2,3)} = \frac{-y^2}{\frac{1}{y} + 2xy - 6}\Big|_{(2,3)} = \frac{-9}{\frac{1}{3} + 2(6) - 6} = \frac{-9}{\frac{19}{3}} = \boxed{\frac{-27}{19}}$$

$\frac{1}{3} + 6$

m = slope of tg. line to the curve at $(2,3)$

$$= y'\Big|_{(2,3)} = -\frac{27}{19}$$

$\Rightarrow$ tg. line eqn.: $y - 3 = \left(-\frac{27}{19}\right)(x-2)$

$$19y - 57 = -27x + 54$$

$$\boxed{27x + 19y = 111}$$

Logarithmic Diff:

$$* \begin{cases} \log_a(A\cdot B) = \log_a A + \log_a B \\ (a>0,\ a\neq 1) \quad \log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B \\ \log_a A^r = r \log_a A \end{cases}$$

$$\frac{d}{dx}\left[\ln(u(x))\right] = \frac{1}{u(x)}\cdot u'(x)$$

eg. $y = \dfrac{(x^3+1)(x^2-1)}{(x^8+6x^4+1)}$, $y' = ?$

$$\ln y = \ln(x^3+1) + \ln(x^2-1) - \ln(x^8+6x^4+1)$$

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{2x}{x^2-1} - \frac{8x^7+24x^3}{x^8+6x^4+1}$$

Ex. 6-10: Find $y'(x) = ?$ for $y = f(x) = \boxed{(x+e^x)^{\ln x}}$

$$\ln y = \ln(x+e^x)^{\ln x} = (\ln x)(\ln(x+e^x))$$

$$\frac{d}{dx} \Rightarrow \boxed{\frac{y'}{y}} = \left(\frac{1}{x}\right)\left(\ln(x+e^x)\right) + (\ln x)\cdot\left(\frac{1}{x+e^x}\cdot(1+e^x)\right) = \frac{\ln(x+e^x)}{x} + \frac{(1+e^x)(\ln x)}{(x+e^x)}$$

$$\Rightarrow y' = y\left[\quad\right] = (x+e^x)^{\ln x}\cdot\left[\frac{\ln(x+e^x)}{x} + \frac{(1+e^x)(\ln x)}{(x+e^x)}\right]$$

6-42 $f(x) = \dfrac{(3x^4+x^2)^6}{(1+x+x^2)^8} \Rightarrow \ln(f(x)) = 6\ln(3x^4+x^2) - 8\ln(1+x+x^2)$

$f'(x) = ?$

$$\frac{f'(x)}{f(x)} = 6\cdot\frac{(12x^3+2x)}{3x^4+x^2} - 8\,\frac{(1+2x)}{1+x+x^2}$$

$$\Rightarrow f'(x) = \left[\frac{(3x^4+x^2)^6}{(1+x+x^2)^8}\right]\cdot\left[\frac{6(12x^3+2x)}{(3x^4+x^2)} - \frac{8(1+2x)}{(1+x+x^2)}\right]$$

6.43 $f(x)=(\ln x)^x$

$$\ln(f(x)) = \ln(\ln x)^x = (x)(\ln(\ln x))$$

$$\frac{d}{dx}: \quad \frac{f'(x)}{f(x)} = (1)\cdot(\ln(\ln x)) + \cancel{(x)}\cdot\left[\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}}\right]$$

$$= \ln(\ln x) + \frac{1}{\ln x}$$

$$\boxed{f'(x) = \underbrace{(\ln x)^x}_{f(x)}\cdot\left[\ln(\ln x)+\frac{1}{\ln x}\right]}$$

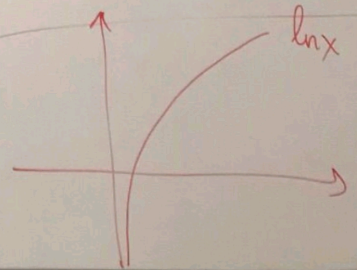
L'Hôpital's rule:

$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty \longrightarrow$ indeterminate forms

If; $\lim\limits_{x\to a}\frac{f(x)}{g(x)} \to \frac{\infty}{\infty}, \frac{0}{0}$, and

suppose that $g'(x)\neq 0$ on an interval containing "$x=a$" $\Rightarrow$

$$\lim_{x\to a}\frac{f(x)}{g(x)} \underset{\text{L.H.}}{=} \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$, if this limit exists or is $\mp\infty$.



Ex. 7.5: $\lim\limits_{x\to\infty}\left(\frac{e^x}{x^3}\right)=\frac{\infty}{\infty}$

$$\overset{L.H.}{\Rightarrow} = \lim_{x\to\infty}\left(\frac{e^x}{3x^2}\right) \underset{\frac{\infty}{\infty}}{\overset{L.H.}{=}} \lim_{x\to\infty}\left(\frac{e^x}{6x}\right) \underset{\frac{\infty}{\infty}}{\overset{L.H.}{=}} \lim_{x\to\infty}\left(\frac{e^x}{6}\right) = \frac{\infty}{6} = \boxed{\infty}$$

Ex. 7.6: $\lim\limits_{x\to1}\left(\frac{x^{10}-1}{x^7-1}\right)=\left(\frac{0}{0}\right)$

$$x^{10}-1=(x-1)(x^9+x^8+\cdots+1)$$

$$x^7-1=(x-1)(x^6+x^5+\cdots+1)$$

$$\lim_{x\to1}\left(\frac{x^{10}-1}{x^7-1}\right)=\lim_{x\to1}\frac{\cancel{(x-1)}\overbrace{(x^9+\cdots+1)}^{10}}{\cancel{(x-1)}\underbrace{(x^6+\cdots+1)}_{7}}=\frac{10}{7}$$

$$\overset{L.H.}{=}\lim_{x\to1}\frac{10x^9}{7x^6}=\frac{10}{7}$$

Ex. 7.7: $\lim\limits_{x\to0}\left(\frac{e^x-1-x-\frac{x^2}{2}}{x^3+x^4}\right)=\left(\frac{0}{0}\right)$

$$\overset{L.H.}{=}\lim_{x\to0}\left(\frac{e^x-1-x}{3x^2+4x^3}\right)=\left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=}\lim_{x\to0}\left(\frac{e^x-1}{6x+12x^2}\right)=\left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=}\lim_{x\to0}\left(\frac{e^x}{6+24x}\right)=\boxed{\frac{1}{6}}$$

Ex. 7.8: $\lim\limits_{x\to\infty}\left(\frac{\ln x+x^2}{xe^x}\right)=\frac{\infty}{\infty}\Rightarrow$ use L.H.

$$\overset{L.H.}{=}\lim_{x\to\infty}\left(\frac{\frac{1}{x}+2x}{e^x+xe^x}\right)=\frac{0+\infty}{\infty}=\frac{\infty}{\infty}$$

$$\overset{L.H.}{=}\lim_{x\to\infty}\left[\frac{\left(-\frac{1}{x^2}+2\right)}{e^x+e^x+xe^x}\right]=\frac{0+2}{\infty}=\frac{2}{\infty}=\boxed{0}$$

7.36 $\lim\limits_{x\to0^+}(x\cdot\ln x)=0\cdot(-\infty)$

$$=\lim_{x\to0^+}\left(\frac{\ln x}{\frac{1}{x}}\right)=\frac{-\infty}{\infty}\to \text{indeterminate form}$$

$\Rightarrow$ L.H.

$$\overset{L.H.}{=}\lim_{x\to0^+}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}=\lim_{x\to0^+}(-x)=\boxed{0}\checkmark$$