Matrix Operations, Matrix Multiplication

$$A = \begin{bmatrix} 1 & 6 & 0 & 4 \\ 2 & 3 & 1 & 0 \\ 4 & 7 & 2 & 8 \end{bmatrix}$$

rows of matrix A; an array of numbers

size(order) of A is 3×4.

columns of matrix A

matrix A has $\boxed{3 \text{ rows}}$ and $\boxed{4 \text{ columns}}$

$\Rightarrow$ matrix A is of size(order) : $\boxed{3}\times\boxed{4}$

$*$ each number in the matrix A is called the $\boxed{\text{entry}}$ at the i^{th} row/j^{th} column position: $\boxed{a_{ij}}$

eg: $a_{23}=1 \quad a_{32}=7\ ,\ a_{34}=8, \ldots$

$A=[a_{ij}]_{3\times 4}$

Let $k\in\mathbb{R}$. Then the multiplication of a matrix $A=[a_{ij}]_{n\times m}$ by a constant $k\in\mathbb{R}$: $k[a_{ij}]_{n\times m}=[ka_{ij}]_{n\times m}$ is determined by multiplying each entry of A by the same constant $k\in\mathbb{R}$:

$$\underset{3\times 4}{3A} = \begin{bmatrix} 3\times1 & 3\times6 & 3\times0 & 3\times4 \\ 3\times2 & 3\times3 & 3\times1 & 3\times0 \\ 3\times4 & 3\times7 & 3\times2 & 3\times8 \end{bmatrix} = \begin{bmatrix} 3 & 18 & 0 & 12 \\ 6 & 9 & 3 & 0 \\ 12 & 21 & 6 & 24 \end{bmatrix}_{3\times4} = [3a_{ij}]_{3\times4}$$

$i=1,2,3$
$j=1,2,3,4$

Sum of two matrices: Matrices of the same order (size) can be added by adding the corresponding entries of the 2 matrices at the same position:

eg. $A=\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{bmatrix}_{2\times3}$ $\quad B=\begin{bmatrix} -1 & 4 & 7 \\ 0 & 5 & -3 \end{bmatrix}_{2\times3}$

$$A+B=\begin{bmatrix} 2+(-1) & 3+4 & 4+7 \\ -1+(0) & 0+5 & 5+(-3) \end{bmatrix}=\begin{bmatrix} 1 & 7 & 11 \\ -1 & 5 & 2 \end{bmatrix}_{2\times3}$$

$$C=\begin{bmatrix} -1 & 0 \\ 4 & 5 \\ 7 & -3 \end{bmatrix}_{3\times2} \quad \left.\begin{matrix} A+C \\ \text{or} \\ B+C \end{matrix}\right\} \text{ are } \underline{\text{not}} \text{ } \underline{\text{defined.}}$$

Difference of matrices:

$$A_{n\times m}-B_{n\times m}=A_{n\times m}+(-B)_{n\times m}$$

$$[a_{ij}]_{n\times m}-[b_{ij}]_{n\times m}=[a_{ij}+(-1)b_{ij}]_{n\times m}$$

eg. $A-B=A+(-1)B \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{bmatrix}+\begin{bmatrix} 1 & -4 & -7 \\ 0 & -5 & 3 \end{bmatrix}=\begin{bmatrix} 3 & -1 & -3 \\ -1 & -5 & 8 \end{bmatrix}$ ✓

eg. Write $A=[a_{ij}]$ if A is 2×2 and $\boxed{a_{ij}=2i+j}$ $\quad i=1,2$; $j=1,2$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 2(1)+1 & 2(1)+2 \\ 2(2)+1 & 2(2)+2 \end{bmatrix}=\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

eg. Write $A=[a_{ij}]$ if A is $\boxed{2\times3}$ and $a_{ij}=(-1)^i-2j$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=\begin{bmatrix} (-1)^1-2(1) & (-1)^1-2(2) & (-1)^1-2(3) \\ (-1)^2-2(1) & (-1)^2-2(2) & (-1)^2-2(3) \end{bmatrix}=\begin{bmatrix} -3 & -5 & -7 \\ -1 & -3 & -5 \end{bmatrix}$$

Equal matrices: Two matrices of equal (same) orders (sizes) are said to be equal $\Leftrightarrow$ the corresponding entries are equal, at the same positions of the two matrices.

eg. Determine the values of x & y if:

$$\begin{bmatrix} x & y+2 \\ x+y & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} x & 1 \\ 0 & 4 \end{bmatrix}_{2\times 2} \Leftrightarrow \quad x=x\checkmark \quad \boxed{y+2=1} \quad \boxed{x+y=0} \quad 4=4\checkmark$$

$$\left.\begin{array}{l} y+2=1 \Rightarrow y=-1 \\ \Rightarrow x+y=0 \Rightarrow x=-y=-(-1)=1 \end{array}\right\} \boxed{(x,y)=(1,-1)}$$

eg. If $\begin{bmatrix} 3x & -y \\ x & y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ x & y \end{bmatrix} \Rightarrow$ then $y=?$

$$3x=3 \quad \left.\begin{array}{l} -y=2 \\ y=y \end{array}\right\} \quad \left.\begin{array}{l} 3x=3 \\ x=x \end{array}\right\} x=1 \qquad \boxed{y=-2}$$

eg. If $4\begin{bmatrix} 1 & x \\ -2 & 0 \end{bmatrix} + 2\begin{bmatrix} -2 & 0 \\ y & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{zero matrix (every entry is 0)}} \Rightarrow \boxed{x=?\ y=?}$

matrix eqn.

$$\begin{bmatrix} 4-4 & 4x \\ -8+2y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 4x \\ -8+2y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} 4x=0 \Rightarrow \boxed{x=0} \\ -8+2y=0 \Rightarrow \boxed{y=4} \\ 2y=8 \end{array}$$

* A square matrix is one in which # of rows and # of columns are equal.

i) Identity matrix: $\begin{cases} a_{ij}=1 & \text{if } i=j \\ a_{ij}=0 & \text{if } i\neq j \end{cases}$ and $i,j=1,\dots,n$

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = I_{2\times 2}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_{3\times 3}$$

2x2 identity matrix

main diagonal entries are 1 in an identity matrix.

ii) Diagonal matrix: is a square matrix for which every entry except the main diagonal are zero

eg. $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

iii) Triangular matrix: there may be some non-zero entries either above or below the main diagonal

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow$ upper-triangular matrix

$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow$ lower triangular matrix

eg. Solve the matrix equation:

$$x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ x-y-3 \end{bmatrix}_{3\times 1} \Rightarrow \begin{bmatrix} x+6 \\ 2x+10+3y \\ 3x+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ x-y-3 \end{bmatrix}$$

column matrices

$x+6=4 \Rightarrow \boxed{x=-2}$

$\Rightarrow 2x+10+3y=3 \Rightarrow 2x+3y=-7 \Rightarrow 2(-2)+3y=-7 \Rightarrow 3y=-3 \Rightarrow \boxed{y=-1}$

Check: $\boxed{x=-2, y=-1}$ in the 3^{rd} eqn.:

$3x+2 \overset{?}{=} x-y-3 \Rightarrow \underbrace{3(-2)+2}_{-4} \overset{?}{=} (-2)-(-1)-3 = -4$ ✓

Soln.: $\boxed{x=-2, y=-1}$

(* if $x=-2$, $y=-1$ didn't satisfy the 3^{rd} eqn.; then we would have no solution.

Transpose of a matrix: the matrix B obtained from $A=[a_{ij}]$ by interchanging the rows and columns of A is called the transpose of A and denoted by A^T:

$$\left.\begin{matrix} A^T = B = [a_{ji}] \\ (A=[a_{ij}]) \end{matrix}\right\} \quad \text{e.g. } A=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

e.g. $A=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$, $B=\begin{bmatrix} 2 & -1 & -2 \\ 1 & -3 & -2 \end{bmatrix}_{2\times 3} \Rightarrow (A+B)^T = ?$

$$A+B=\begin{bmatrix} 3 & 1 & 1 \\ 4 & 1 & 3 \end{bmatrix}_{2\times 3} \Rightarrow (A+B)^T = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2}$$

$$A^T=\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}, \quad B^T=\begin{bmatrix} 2 & 1 \\ -1 & -3 \\ -2 & -2 \end{bmatrix} \Rightarrow A^T+B^T=\begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = (A+B)^T$$

$\Rightarrow \boxed{(A+B)^T = A^T + B^T}$ if A & B are the same size.

e.g. $$\left(\underbrace{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{3\times 3} + \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}_{2\times 3}}_{\text{not poss.}} \right)^T \rightarrow \text{undefined}$$

Trace of a ~~diagonal~~ square matrix: $\boxed{\text{trace}(A)}$ is the sum of the diagonal entries.

e.g. $\text{trace}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} = 1+(-1)+(-4) = -4$

e.g. $\text{trace}\overbrace{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}}^{B} = 3-2+2 = 3$

$\text{trace}\left[B^T\right] = \text{trace}\begin{bmatrix} 3 & 0 & 3 \\ -2 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 3-2+2 = 3$

$\boxed{\text{trace}(A) = \text{trace}(A^T)}$