Outline for Math 113/M-2 :

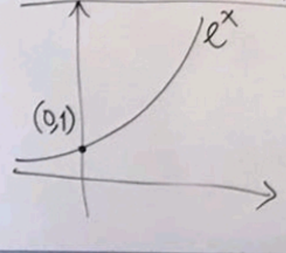
(21/12/2021 → 17:30/L111)

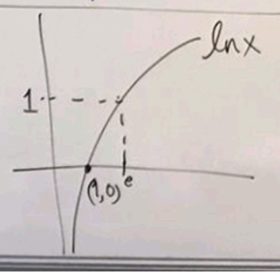
[out of 110 pts. → 10 pts. Bonus]

Differentiation:

100 + 10 pts.

- Der. of polyns., rat. funcs., exp, logs
- Chain rule
- Implicit diff.
- Logarithmic diff.
- L'Hôpital's rule
- ~~Applied Optimization~~





$\overbrace{\text{L'Hôpital's rule}}^{\text{L.H.}}$: If

$$\boxed{\lim_{x\to\mp\infty}\frac{f(x)}{g(x)}=\begin{cases}0/0\\ \infty/\infty\end{cases}} \text{ or } \boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\begin{cases}0/0\\ \frac{\infty}{\infty}\end{cases}} \text{ and if } g'(a)\neq 0 \text{ (a can be } \mp\infty)$$

$$\Rightarrow \boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}} \text{ if this limit exists or is } \mp\infty.$$

7.22 $\lim_{x\to\infty}\left(\frac{3x^2+4\ln x}{6x^2+7\ln x}\right)=\frac{\infty}{\infty}\Rightarrow$ use L.H.

$$\overset{\text{L.H.}}{=}\lim_{x\to\infty}\frac{6x+4\left(\frac{1}{x}\right)}{12x+7\left(\frac{1}{x}\right)}\underset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to\infty}\frac{\frac{6x^2+4}{\cancel{x}}}{\frac{12x^2+7}{\cancel{x}}}=\frac{\infty}{\infty}\overset{\text{L.H.}}{\Longrightarrow}\lim_{x\to\infty}\frac{12\cancel{x}+0}{24x+0}=\frac{12}{24}=\frac{1}{2}$$

or

$$\overset{\text{L.H.}}{=}\lim_{x\to\infty}\frac{6-\frac{4}{x^2}}{12-\frac{7}{x^2}}=\frac{6-0}{12-0}=\frac{1}{2}$$

$(a,b,c \neq 0)$

7.24

$$\lim_{x\to\infty} \frac{\ln(x+x^4)}{x} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1+4x^3}{x+x^4}}{1} = \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left[\frac{1}{x^3}+4\right]}{\cancel{x^3}\left[\frac{1}{x^2}+x\right]} = \frac{0+4}{0+\infty} = \frac{4}{\infty} = \boxed{0}$$

7.28

$$\lim_{x\to e} \left(\frac{\ln x - 1}{x-e}\right) = \left(\frac{0}{0}\right) \quad [\ln e = 1]$$

$$\overset{L.H.}{=} \lim_{x\to e} \left(\frac{\frac{1}{x}}{1}\right) = \boxed{\frac{1}{e}}$$

7.33

$$\lim_{x\to 0} \left\{\frac{\sqrt{a+bx} - \sqrt{a+cx}}{x}\right\} = \left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{x\to 0} \frac{\frac{b}{2\sqrt{a+bx}} - \frac{c}{2\sqrt{a+cx}}}{1}$$

$$= \frac{b}{2\sqrt{a}} - \frac{c}{2\sqrt{a}} = \boxed{\frac{b-c}{2\sqrt{a}}}$$

HW: do the same limit using multiplication & division by the conjugate of the numerator.

7.27

$$\lim_{x\to \frac{1}{2}} \left[\frac{\ln(2x)}{2x^2+x-1}\right] = \frac{(\ln 1) = 0}{\underbrace{2\left(\frac{1}{4}\right)+\frac{1}{2}-1}_{0}} = \left(\frac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{x\to \frac{1}{2}} \left(\frac{\frac{1}{\cancel{2}x}\cdot(\cancel{2})}{4x+1}\right) = \frac{\frac{1}{\frac{1}{2}}}{4\left(\frac{1}{2}\right)+1} = \frac{2}{\frac{3}{2}} = \boxed{\frac{4}{3}}$$

7.31

$$\lim_{x\to\infty} \left(\frac{\ln x}{\sqrt[3]{x}}\right) = \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}}$$

$$= \lim_{x\to\infty} \frac{3x^{2/3}}{x} = \lim_{x\to\infty} \boxed{\frac{3}{x^{1/3}}}$$

$$= \frac{3}{\infty} = \boxed{0}$$

$$\left(x^{1/3}\right)' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\left[(a+bx)^{1/2}\right]' = \frac{1}{2}(a+bx)^{-1/2}\cdot(b)$$

7.34 $\lim\limits_{x\to 0}\left(\frac{4^x-1}{2^x-1}\right) = \frac{1-1}{1-1} = \left(\frac{0}{0}\right) \left[4^0=1,\ 2^0=1\right]$

$\overset{L.H.}{=} \lim\limits_{x\to 0}\frac{(\ln 4)(4^x)}{(\ln 2)(2^x)} = \frac{\ln 4}{\ln 2} = \frac{\ln 2^2}{\ln 2} = \frac{2\cancel{\ln 2}}{\cancel{\ln 2}} = \boxed{2}$

7.35 $\lim\limits_{x\to 2}\left[\frac{\ln\left(\frac{x}{2}\right)}{\underbrace{x(x-2)}_{(x^2-2x)}}\right] = \boxed{\left(\frac{0}{0}\right)}$

$\overset{L.H.}{=} \lim\limits_{x\to 2}\frac{\frac{1}{\left(\frac{x}{2}\right)}\cdot\left(\frac{1}{2}\right)}{(2x-2)} = \lim\limits_{x\to 2}\left[\frac{1}{2x(x-1)}\right]$

$= \frac{1}{4(1)} = \boxed{\frac{1}{4}}$

$\boxed{b>0,\ b\neq 1}$ $\begin{cases} (\log_b x)' = \left(\frac{\ln x}{\ln b}\right)' = \boxed{\frac{1}{\ln b}\cdot\frac{1}{x}} \\ (b^x)' = b^x\cdot(\ln b) \end{cases}$

$\boxed{y=b^x} \Rightarrow \boxed{\ln y} = \ln b^x = x\cdot(\ln b)$

log. diff. $\}$ $\frac{y'}{y} = (\ln b)(1) \Rightarrow y' = (\ln b)\,y$

$\boxed{y' = (\ln b)(b^x)}$

$\boxed{\begin{array}{l} \ln(u(x)) = \frac{1}{u(x)}\cdot u'(x) = \frac{u'}{u} \\[2ex] b^{u(x)} = b^{u(x)}\cdot u'(x)\cdot \ln b \end{array}}$

Using $\overbrace{\text{implicit diff.}}^{y'(x)}$: $\Rightarrow$

$$3y^2 \cdot y' = 2x - \frac{(1\cdot y + x\cdot y')}{xy+1}$$

$$y'\left[3y^2 + \frac{x}{xy+1}\right] = 2x - \frac{y}{xy+1}$$

$$y'\Big|_{(0,-2)} = \frac{2x - \frac{y}{xy+1}}{3y^2 + \frac{x}{xy+1}}\Bigg|_{(0,-2)} = m = \frac{1}{6}$$

↳ slope of tg. line

$\Rightarrow$ tg. line: $y-(-2) = \frac{1}{6}(x-0)$

2) a) $y^x = x^{y^2}$, $y' = ?$

$$\boxed{\ln y^x = \ln x^{y^2}} \Rightarrow \boxed{x\cdot \ln y = y^2\cdot \ln x} \xrightarrow{\text{product rule}} \left(1\cdot \ln y + x\cdot \frac{1}{y}\cdot y'\right) = (2y\,y')(\ln x) + y^2\left(\frac{1}{x}\right)$$

$$y'\left[\frac{x}{y} - 2y\ln x\right] = \frac{y^2}{x} - \ln y \Rightarrow y' = \frac{\frac{y^2 - x\ln y}{x}}{\frac{x - 2y^2\ln x}{y}} = \frac{y^2 - x\ln y}{x}\cdot\frac{y}{x - 2y^2\ln x}$$

$$\boxed{y' = \frac{y^3 - xy\ln y}{x^2 - 2xy^2\ln x}}$$

b) $f(x) = \log_{x^2}\sqrt{x^2-5x}$, $f'(x) = ?$

$$f(x) = \frac{\ln(x^2-5x)^{1/2}}{\ln x^2} = \frac{\frac{1}{2}\ln(x^2-5x)}{2\ln x} = \left(\frac{1}{4}\right)\left[\frac{\ln(x^2-5x)}{\ln x}\right]$$

change of log base

$$f'(x) = \frac{1}{4}\left[\frac{\left(\frac{1}{x^2-5x}(2x-5)\right)(\ln x) - \left(\frac{1}{x}\right)\left(\ln(x^2-5x)\right)}{(\ln x)^2}\right]$$

quotient rule

2016/MII/Q2

$$f(x)=\begin{cases} x^2-3x+5, & \text{if } x<-2 \\ 3, & \text{if } x=-2 \\ x^3+2x^2-5x-7, & \text{if } x>-2 \end{cases}$$ Find $f'(-2)$ if it exists!

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad \boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

deriv. at $x=a$

$$\boxed{f'(-2)}=\lim_{x\to -2}\left[\frac{f(x)-f(-2)}{x-(-2)}\right]=\lim_{x\to -2}\left[\frac{f(x)-3}{x+2}\right]$$

$$\boxed{\lim_{x\to -2^+}(\quad)}=-1$$

$$\boxed{\lim_{x\to -2^-}(\quad)}=\text{d.n.e.}$$

$$\not= \Rightarrow \lim_{x\to -2}\underbrace{\left[\frac{f(x)-f(-2)}{x-(-2)}\right]}_{f'(-2)}=\text{d.n.e.}$$

$$\lim_{\substack{x\to -2^+ \\ (x>-2)}}(\quad)=\lim_{\substack{x\to -2^+ \\ (x>-2)}}\frac{(x^3+2x^2-5x-7)-3}{x+2}=\lim_{x\to -2^+}\left[\frac{x^3+2x^2-5x-10}{x+2}\right]\left(\frac{0}{0}\right)$$

$$\overset{\text{L.H.}}{=}\lim_{x\to -2^+}\left[\frac{3x^2+4x-5}{1}\right]=\frac{-1}{1}=-1$$

$$\lim_{\substack{x\to -2^- \\ (x<-2)}}(\quad)=\lim_{x\to -2^-}\left[\frac{(x^2-3x+5)-(3)}{x+2}\right]=\lim_{x\to -2}\frac{(x^2-3x+2)}{x+2}$$

$$=\frac{12}{0}=\boxed{\infty}\ (\text{d.n.e.})$$

$$\Longrightarrow \boxed{f'(-2)=\text{d.n.e.}}$$

Q: $f(1)=0,\ f'(1)=3,\ f(2)=-1,\ f'(2)=-2$
$g(1)=4,\ g'(1)=2,\ g(2)=1,\ g'(2)=5$

a) If $\boxed{h(x)=\dfrac{2+f(x)}{g(x)-x^2+1}} \Rightarrow h'(1)=?$

$$h'(x)=\frac{[f'(x)][g(x)-x^2+1]-[g'(x)-2x][2+f(x)]}{(g(x)-x^2+1)^2}$$

quotient rule

$$h'(1)=\frac{(f'(1))(g(1)-\cancel{1^2}+\cancel{1})-(g'(1)-2(1))(2+f(1))}{(g(1)-\cancel{1^2}+\cancel{1})^2}$$

$$=\frac{(3)(4)-\cancel{(2-2)}(2+0)}{(4)^2}$$

$$=\frac{12}{16}=\boxed{\frac{3}{4}}$$

b) $k(x)=f(g(2x))=(f\circ g)(2x) \Rightarrow k'(1)=?$

Using chain rule;

$k'(x)=f'(g(2x))\cdot g'(2x)\cdot(2)$

$k'(1)=f'(g(2))\cdot g'(2)\cdot 2$

$=\underbrace{f'(1)}_{3}\cdot(5)(2)=3\cdot 5\cdot 2=\boxed{30}$

Fall 2020/M II (Math 113)

Exam Archive (?) (Review Probs.)

※1) Eqn. of tg. line to $\boxed{y^3=x^2-\ln(xy+1)-8}$ at the pt. where $\boxed{x_0=0}$

Soln.:

$x_0=0 \Rightarrow y_0^3=0-\ln(\underset{0}{\cancel{0}}+1)-8=-8$

$y^3=-8 \Rightarrow \boxed{y_0=-2} \Longrightarrow \boxed{\text{at the pt. } (0,-2)}$