Math 113 - Midterm II Solutions

1-) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2\le x<3 \\ 2x-a+b & \text{if } x>3 \end{cases}$

at $x=2$:

$$\lim_{\substack{x\to 2^- \\ (x<2)}} f(x) = \lim_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 2^-}\left(\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}\right) = 2+2 = 4$$

$$\lim_{\substack{x\to 2^+ \\ (x>2)}} f(x) = \lim_{x\to 2^+}(ax^2-bx+3) = \boxed{4a-2b+3}$$

For continuity at $x=2$; $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \Rightarrow$

$$4a-2b+3=4$$

$$\Rightarrow \boxed{4a-2b=1} \; (*)$$

at $x=3$: $\lim_{\substack{x\to 3^- \\ (x<3)}} f(x) = \lim_{x\to 3^-}(ax^2-bx+3) = \boxed{9a-3b+3}$

$$\lim_{\substack{x\to 3^+ \\ (x>3)}} f(x) = \lim_{x\to 3^+}(2x-a+b) = \boxed{6-a+b}$$

For continuity at $x=3$; $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \Rightarrow$

$$9a-3b+3 = 6-a+b$$

$$\Rightarrow \boxed{10a-4b=3} \; (**)$$

Solving $(*)$ and $(**)$ simultaneously, we get:

$4a-2b=1 \; (\times(-2)) \Rightarrow -8a+\cancel{4b}=-2$

$10a-4b=3 \qquad\qquad\quad\;\; 10a-\cancel{4b}=3$

$$2a=1 \Rightarrow \boxed{a=\frac{1}{2}}$$

From $(*)$: $2b = 4a-1$

$= 4\left(\frac{1}{2}\right)-1 = 1$

$\Rightarrow \boxed{b=\frac{1}{2}}$

$\boxed{\text{Soln: } a=\frac{1}{2},\; b=\frac{1}{2}}$

2.) a) $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^3-3x^2+3x-1}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 1}\frac{\cancel{(x-1)}(x-2)}{(x-1)^{\cancel{3}\,2}} = \lim_{x\to 1}\frac{(x-2)}{(x-1)^2}$

$= \frac{-1}{0} = \boxed{-\infty}$

(d.n.e)

b.) $\lim_{x\to -\infty}\left(\sqrt{x^2+x+6}+x+2\right) = \infty - \infty$

$= \lim_{x\to -\infty}\frac{\left(\sqrt{x^2+x+6}+(x+2)\right)\left(\sqrt{x^2+x+6}-(x+2)\right)}{\left(\sqrt{x^2+x+6}-(x+2)\right)}$

$\left(\lim_{x\to -\infty}\sqrt{x^2} = -x\right)$ $= \lim_{x\to -\infty}\left[\frac{(x^2+x+6)-(x+2)^2}{\left(\sqrt{x^2\left(1+\frac{1}{x}+\frac{6}{x^2}\right)}-x-2\right)}\right]$

$(2-3x)$

$= \lim_{x\to -\infty}\frac{\cancel{x^2}+x+6-\cancel{x^2}-4x-4}{-x\sqrt{1+\frac{1}{x}+\frac{6}{x^2}}-x-2} = \lim_{x\to -\infty}\frac{\cancel{-x}\left(-\frac{2}{x}+3\right)}{\cancel{-x}\left(\sqrt{1+\frac{1}{x}+\frac{6}{x^2}}+1+\frac{2}{x}\right)}$

$= \lim_{x\to -\infty}\left(\frac{-\frac{2}{x}+3}{\sqrt{1+\frac{1}{x}+\frac{6}{x^2}}+1+\frac{2}{x}}\right) = \frac{0+3}{\sqrt{1+0+0}+1+0} = \boxed{\frac{3}{2}}$

3.) $f(x) = \frac{-4}{2-3x}$, $f(x+h) = \frac{-4}{2-3(x+h)} = \frac{-4}{2-3x-3h}$

$\boxed{f'(x)} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{-4}{2-3x-3h}-\frac{-4}{2-3x}}{h}$

$= \lim_{h\to 0}\left[\frac{-4(2-3x)-(-4)(2-3x-3h)}{h(2-3x-3h)(2-3x)}\right] = \lim_{h\to 0}\left[\frac{\cancel{-8}+\cancel{12x}+\cancel{8}-\cancel{12x}-12h}{h(2-3x-3h)(2-3x)}\right]$

$= \lim_{h\to 0}\left[\frac{-12\cancel{h}}{\cancel{h}(2-3x-3h)(2-3x)}\right] = \boxed{\frac{-12}{(2-3x)^2}}$

4-) $x_0=2 \Rightarrow f(2)=e^{(2^3-4(2)^2+2(2)+4)}=e^0=\boxed{1} \Rightarrow (x_0,y_0)=(2,1)$

$f'(x)=\left(e^{(x^3-4x^2+2x+4)}\right)(3x^2-8x+2)$

$f'(2)=e^0\cdot(3(2)^2-8(2)+2)=(1)\cdot(-2)=\boxed{-2} \rightarrow$ slope of tg line

tg. line eqn.: $y-1=(-2)(x-2)$

$y-1=-2x+4 \Rightarrow \boxed{y=-2x+5}$

5-) a-) $f(x)=\dfrac{\ln x}{e^x} \Rightarrow f'(x)=\dfrac{(1/x)(e^x)-(e^x)\cdot\ln x}{(e^x)^2}$

$\Rightarrow f'(x)=\dfrac{e^x-x\cdot e^x\ln x}{x\cdot e^{2x}}=\dfrac{\cancel{e^x}(1-x\ln x)}{x\cdot e^x\cancel{e^x}}$

$\Rightarrow \boxed{f'(x)=\dfrac{1-x\ln x}{x\cdot e^x}}$

b-) $f(x)=\dfrac{x^2+1}{xe^x} \Rightarrow f'(x)=\dfrac{(2x)(xe^x)-(x^2+1)(e^x+xe^x)}{(x\cdot e^x)^2}$

$\Rightarrow f'(x)=\dfrac{2x^2e^x-x^2e^x-x^3e^x-e^x-xe^x}{x^2\cdot e^{2x}}=\dfrac{\cancel{e^x}(x^2-x^3-x-1)}{x^2\cdot e^x\cancel{e^x}}$

$=\boxed{\dfrac{x^2-x^3-x-1}{x^2\cdot e^x}}$

6-) a) $3x^2y^2+2xy^2+y^2+x^3=0 \qquad y'=?$

$(6xy^2+3x^2\cdot 2y\cdot y')+2(1\cdot y^2+x\cdot 2y\,y')+2y\cdot y'+3x^2=0$

$y'(6x^2y+4xy+2y)=-3x^2-6xy^2-2y^2$

$\Rightarrow \boxed{y'=-\dfrac{3x^2+6xy^2+2y^2}{4xy+6x^2y+2y}}$

b.) $\overbrace{\ln y^x}^{x\cdot\ln y}$ in $x^3y + \ln y^x + e^{x^2y} = 3$, $\quad y' = ?$

$$(3x^2y + x^3\cdot y') + \left(1\cdot\ln y + x\cdot\frac{y'}{y}\right) + e^{x^2y}(2xy + x^2\cdot y') = 0$$

$$y'\left(x^3 + \frac{x}{y} + e^{x^2y}x^2\right) = -3x^2y - \ln y - e^{x^2y}\cdot 2xy$$

$$y' = \frac{-3x^2y - \ln y - e^{x^2y}\cdot 2xy}{x^3 + \frac{x}{y} + e^{x^2y}\cdot x^2} = \boxed{-\frac{3x^2y^2 + y\cdot\ln y + 2xy^2\cdot e^{x^2y}}{yx^3 + x + x^2y\cdot e^{x^2y}}}$$

Bonus Question: $h(1)=2,\ g(2)=3,\ g'(2)=5,\ h'(1)=4,\ f'(3)=6$

$$k(x) = f(g(h(x^3)))$$

$$k'(x) = f'(g(h(x^3)))\cdot g'(h(x^3))\cdot h'(x^3)\cdot 3x^2$$

$$k'(1) = f'(g(h(1)))\cdot g'(h(1))\, h'(1)\cdot 3(1)^2$$

$$= f'(g(2))\cdot g'(2)\cdot 4\cdot 3$$

$$= f'(3)\cdot 5\cdot 4\cdot 3 = 6\cdot 5\cdot 4\cdot 3 = \boxed{360}$$