ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113 - Mathematics for City Planners

2021-2022 Fall

SECOND MIDTERM EXAMINATION

21.12.2021, 17:30

SOLUTIONS

Question	Grade	Out of
1		24
2		30
3		28
4		16
5		12
Total		110

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Find the derivative of the following functions. Simplify your answers as much as possible.

a) (12 pts.) $f(x) = \ln\left[\left(\frac{x^2+5x+4}{x^3+2}\right)^6\right]$

$$f(x) = 6\left[\ln(x^2+5x+4) - \ln(x^3+2)\right]$$

$$f'(x) = 6\left[\frac{(2x+5)}{x^2+5x+4} - \frac{3x^2}{x^3+2}\right]$$

$$= \boxed{\frac{(12x+30)}{x^2+5x+4} - \frac{18x^2}{x^3+2}}$$

$$\frac{\ln(1-x)^{1/2}}{\ln 4} = \frac{1}{2\ln 4}\ln(1-x)$$

b) (12 pts.) $f(x) = xe^{-x^2} + 2^{x^2+x} + \log_4\sqrt{1-x}$

$$f'(x) = \left[1.e^{-x^2} + x.e^{-x^2}(-2x)\right] + 2^{x^2+x}.(2x+1)(\ln 2) + \frac{1}{2\ln 4}.\left(\frac{(-1)}{1-x}\right)$$

$$\Rightarrow f'(x) = \boxed{e^{-x^2}\left[1-2x^2\right] + 2^{x^2+x}.(2x+1)(\ln 2) - \frac{1}{2\ln 4(1-x)}}$$

2) a) (10 pts.) Let $f(x) = \begin{cases} x^2 - 7x + 20 & \text{if } x < 2 \\ 10 & \text{if } x = 2 \\ -6\sqrt{2x} + 22 & \text{if } x > 2 \end{cases}$. Find $f'(2)$ if it exists.

left deriv. at $x=2$:

$$\lim_{x\to 2^-}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to 2^-}\left[\frac{(x^2-7x+20)-10}{x-2}\right] = \lim_{x\to 2^-}\frac{\cancel{(x-2)}(x-5)}{\cancel{(x-2)}} = -3$$

right deriv. at $x=2$:

$$\lim_{x\to 2^+}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to 2^+}\left[\frac{(-6\sqrt{2x}+22)-10}{x-2}\right] = \lim_{x\to 2^+}\left[\frac{-6(\sqrt{2x}-2)}{x-2}\right]$$

$$= \lim_{x\to 2^+}\left[\frac{-6(\sqrt{2x}-2)(\sqrt{2x}+2)}{(x-2)(\sqrt{2x}+2)}\right] = \lim_{x\to 2^+}\frac{-6\,(2x-4)=2\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{2x}+2)} = \frac{-12}{2+2} = -3$$

$\Rightarrow$ since right and left derivatives at $x=2$ are the same $\Rightarrow$ $\boxed{f'(2)=3}$

b) (10 pts.) Find $f'(x)$ if $f(x) = e^{x^3} + \ln(2x^2+3)$.

$$f'(x) = e^{x^3}\cdot(3x^2) + \frac{4x}{2x^2+3} = \boxed{3x^2\cdot e^{x^3} + \frac{4x}{2x^2+3}}$$

c) (10 pts.) Find y' if $x^3y + e^{xy^2} = 5$.

$$3x^2y + x^3y' + e^{xy^2}\cdot(1.y^2 + x\cdot 2y\cdot y') = 0$$

$$y'(x^3 + 2xye^{xy^2}) = -3x^2y - y^2e^{xy^2}$$

$$\Rightarrow \boxed{y' = -\frac{3x^2y + y^2e^{xy^2}}{x^3 + 2xye^{xy^2}}}$$

3) a) (14 pts.) Find the equation of the tangent line to the curve $2x - 4y^4 + x^2y^6 + 11y^3 = 0$ at $(3, -1)$.

diff. w.r.t. x $\Rightarrow 2 - 16y^3 \cdot y' + (2x \cdot y^6 + x^2 \cdot 6y^5 \cdot y') + (33y^2 \cdot y') = 0$

$y'(-16y^3 + 6x^2y^5 + 33y^2) = -2 - 2xy^6$

$$\Rightarrow \boxed{y' = -\frac{2+2xy^6}{-16y^3+6x^2y^5+33y^2}}$$

$$\Rightarrow y'\Big|_{(3,-1)} = -\frac{2+2(3)(-1)^6}{-16(-1)^3+6(3)^2(-1)^5+33(-1)^2} = -\frac{8}{16-54+33} = \boxed{\frac{8}{5}}$$

slope of tg. line

tg. line eqn.: $y-(-1) = \left(\frac{8}{5}\right)(x-3) \Rightarrow 5y+5 = 8x-24$

$\boxed{8x - 5y = 29}$

b) (14 pts.) Let the function

$$f(x) = \frac{(1+x^3)^7\,(2+x^5)^8}{(3+x^2)^4}$$

be given. Find derivative of the function $f(x)$ at $x = 0$.
(Hint: Use logarithmic differentiation)

$\ln f(x) = 7\ln(1+x^3) + 8\ln(2+x^5) - 4\ln(3+x^2) \Rightarrow$ differentiate

$$\Rightarrow \frac{f'(x)}{f(x)} = 7 \cdot \frac{3x^2}{1+x^3} + 8 \cdot \frac{5x^4}{2+x^5} - 4 \cdot \frac{2x}{3+x^2}$$

$$\Rightarrow \boxed{f'(x) = \left[\frac{(1+x^3)^7 \cdot (2+x^5)^8}{(3+x^2)^4}\right] \cdot \left[\frac{21x^2}{1+x^3} + \frac{40x^4}{2+x^5} - \frac{8x}{3+x^2}\right]}$$

4) Evaluate the following limits:

a) (8 pts.) $\lim_{x\to 3} \frac{x^3-4x-15}{x^2+x-12} = \frac{(3)^3-4(3)-15}{(3)^2+(3)-12} = \left(\frac{0}{0}\right) \Rightarrow$

$$\overset{L.H.}{=} \lim_{x\to 3}\left(\frac{3x^2-4}{2x+1}\right) = \frac{3(3)^2-4}{2(3)+1} = \boxed{\frac{23}{7}}$$

b) (8 pts.) $\lim_{x\to 0} \frac{e^{3x}-1}{\ln(x+1)} = \frac{e^0-1}{\ln 1} = \left(\frac{0}{0}\right) \Rightarrow$

$$\overset{L.H.}{=} \lim_{x\to 0}\left(\frac{3e^{3x}}{\frac{1}{x+1}}\right) = \frac{3(1)}{1} = \boxed{3}$$

5) Given that $f(0) = 2$, $f'(0) = 5$, $f'(3) = 5$, $g(0) = 3$, $g'(0) = 7$, $g'(2) = 10$;

a) (3 pts.) If $h(x) = f(x)g(x)$, then compute $h'(0)$.

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$
$$h'(0) = f'(0)\cdot g(0) + f(0)\cdot g'(0)$$
$$= (5)(3) + (2)(7) = 15 + 14 = \boxed{29}$$

b) (3 pts.) If $k(x) = \dfrac{f(x)}{g(x)}$, then compute $k'(0)$.

$$k'(x) = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2}$$

$$k'(0) = \frac{f'(0)\cdot g(0) - g'(0)\cdot f(0)}{(g(0))^2} = \frac{(5)(3)-(7)(2)}{(3)^2} = \boxed{\frac{1}{9}}$$

c) (6 pts.) If $s(x) = f(g(x))$ and $t(x) = g(f(x))$ then compute $s'(0)$ and $t'(0)$.

$$s'(x) = f'(g(x))\cdot g'(x)$$
$$s'(0) = f'(g(0))\cdot g'(0) = f'(3)\cdot 7 = 5\cdot 7 = \boxed{35}$$

$$t'(x) = g'(f(x))\cdot f'(x)$$
$$t'(0) = g'(f(0))\cdot f'(0) = g'(2)\cdot 5 = 10.5 = \boxed{50}$$