ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113 - Mathematics for City Planners

2022-2023 Fall

SECOND MIDTERM EXAMINATION

22.12.2022, 17:30

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

Question	Grade	Out of
1		24
2		16
3		16
4		30
5		27
Total		113

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Find the derivative of the following functions. Simplify your answers as much as possible.

a) (12 pts.) $f(x) = e^{(3x^4+1)^3}$

$$f'(x) = e^{(3x^4+1)^3} \cdot \left(3(3x^4+1)^2\right) \cdot (12x^3)$$

$$= \boxed{36x^3(3x^4+1)^2 \cdot e^{(3x^4+1)^3}}$$

b) (12 pts.) $f(x) = \log_x \sqrt{x^2 - 4x}$ $= \dfrac{\ln (x^2-4x)^{1/2}}{\ln x} = \dfrac{1}{2} \dfrac{\ln(x^2-4x)}{\ln x}$

$$f'(x) = \frac{1}{2} \frac{\left\{\left(\frac{2x-4}{x^2-4x}\right)(\ln x) - \left(\frac{1}{x}\right)\ln(x^2-4x)\right\}}{(\ln x)^2}$$

$$= \frac{\left(\frac{1}{\cancel{2}} \frac{\cancel{2}(x-2)}{x^2-4x}\right)(\ln x) - \left(\frac{1}{2x}\right)\left(\ln(x^2-4x)\right)}{(\ln x)^2}$$

$$= \boxed{\frac{\left(\frac{x-2}{x^2-4x}\right)\left(\frac{1}{\ln x}\right) - \left(\frac{1}{2x}\right)\left(\ln(x^2-4x)\right)}{(\ln x)^2}}$$

2) (16 pts.) Find the points of discontinuity (if any) of the function

$$f(x) = \begin{cases} e^x + 3 - x^2, & \text{if } x < 0 \\ 3x - 2, & \text{if } 0 \leq x < 1 \\ 2 - x^2 + \ln x, & \text{if } 1 \leq x \end{cases}$$

Explain your answer in detail.

at $x=0$: $f(0) = 3(0) - 2 = -2$

$$\lim_{\substack{x \to 0^- \\ (x<0)}} f(x) = \lim_{x \to 0^-} (e^x + 3 - x^2) = e^0 + 3 - 0 = 1 + 3 = 4$$

$$\lim_{\substack{x \to 0^+ \\ (x>0)}} f(x) = \lim_{x \to 0^+} (3x - 2) = -2$$

$\Rightarrow \lim_{x \to 0} f(x) =$ d.n.e.

$\Rightarrow f$ is discontinuous at $x=0$.

at $x=1$: $f(1) = 2 - 1^2 + \underbrace{\ln(1)}_{0} = 1$

$$\lim_{\substack{x \to 1^- \\ (x<1)}} f(x) = \lim_{x \to 1^-} (3x - 2) = 3 - 2 = 1$$

$$\lim_{\substack{x \to 1^+ \\ (x>1)}} f(x) = \lim_{x \to 1^+} (2 - x^2 + \ln x) = 1$$

$\lim_{x \to 1} f(x) = 1 = f(1)$

$\Rightarrow f$ is cont. at $x=1$.

At all other pts.; $\{e^x + 3 - x^2, 3x - 2, 2 - x^2 + \ln x\}$ are cont. wherever they are defined.

So; the only discontinuity pt. of f is $x=0$

3) (16 pts.) Let $f(x) = \begin{cases} ax - 6 & \text{if } x < 2, \\ b & \text{if } x = 2, \\ x^2 - 4 & \text{if } x > 2. \end{cases}$

Find the values of a and b, so that the function $f(x)$ is continuous at $x = 2$.

$f(2) = b$

$$\left.\begin{aligned} &\lim_{\substack{x\to 2^-\\(x<2)}} f(x) = \lim_{x\to 2^-}(ax-6) = 2a-6 \\ &\lim_{\substack{x\to 2^+\\(x>2)}} f(x) = \lim_{x\to 2^+}(x^2-4) = 4-4=0 \end{aligned}\right\} \lim_{x\to 2} f(x) = 2a-6=0$$

$$\Rightarrow \boxed{a=3}$$

For continuity at $x=2$: $\underbrace{f(2)}_{\substack{\shortparallel\\ b}} = \lim_{x\to 2} f(x) = 0$

$$\Rightarrow \boxed{b=0}$$

4) a) (15 pts.) Find the equation of the tangent line to the curve $y^3 = x^2 - \ln(xy+1) - 8$ at $(0,-2)$.

$$3y^2 \cdot y' = 2x - \frac{1}{xy+1}(1 \cdot y + x \cdot y')$$

$$y'\left(3y^2 + \frac{x}{xy+1}\right) = 2x - \frac{y}{xy+1}$$

$$y'\left(\frac{3xy^3+3y^2+x}{\cancel{xy+1}}\right) = \frac{2x^2y+2x-y}{\cancel{xy+1}} \Rightarrow \boxed{y' = \frac{2x^2+2x-y}{3xy^3+3y^2+x}}$$

$$y'\Big|_{(0,-2)} = \frac{0+0-(-2)}{0+3(-2)^2+0} = \frac{2}{12} = \boxed{\frac{1}{6}} \rightarrow \text{slope of tangent line} \Big\} \Rightarrow y-(-2) = \frac{1}{6}(x-0)$$

$$6y+12 = x$$

$$\Rightarrow \boxed{x - 6y = 12}$$

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b) (15 pts.) Given that $y = x^{x^2+1}$, use logarithmic differentiation to evaluate $y'(1)$=?

$$\ln y = \ln x^{x^2+1} = (x^2+1)(\ln x)$$

$$\frac{y'}{y} = (2x)(\ln x) + (x^2+1)\left(\frac{1}{x}\right) = 2x \cdot \ln x + x + \frac{1}{x}$$

$$\Rightarrow y' = \left(x^{x^2+1}\right)\left(2x \cdot \ln x + x + \frac{1}{x}\right)$$

$$y'(1) = \left(1^{1+1}\right)\left(2(1) \cdot (\underbrace{\ln 1}_{0}) + 1 + 1\right) = \boxed{2}$$

5) Given that $f(0)=2$, $g(0)=3$, $f'(0)=5$, $g'(0)=7$, $f'(3)=\pi$ and $g'(2)=\pi^2$;

a) (9 pts.) If $h(x)=f(x)g(x)$, then compute $h'(0)$?

$$h'(x)=f'(x)\cdot g(x)+f(x)\cdot g'(x)$$
$$h'(0)=f'(0)\cdot g(0)+f(0)\cdot g'(0)$$
$$=(5)(3)+(2)(7)=15+14=\boxed{29}$$

b) (9 pts.) If $k(x)=\dfrac{f(x)}{g(x)}$, then compute $k'(0)$?

$$k'(x)=\frac{f'(x)\cdot g(x)-g'(x)\cdot f(x)}{[g(x)]^2}$$
$$k'(0)=\frac{f'(0)\cdot g(0)-g'(0)\cdot f(0)}{[g(0)]^2}=\frac{(5)(3)-(7)(2)}{(3)^2}$$
$$=\boxed{\frac{1}{9}}$$

c) (9 pts.) If $s(x)=f(g(x))$, then compute $s'(0)$?

$$s'(x)=f'(g(x))\cdot g'(x)$$
$$s'(0)=f'(g(0))\cdot g'(0)$$
$$=f'(3)\cdot 7=(\pi)(7)=\boxed{7\pi}$$