ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113 - Mathematics for City Planners

2022-2023 Fall

FIRST MIDTERM EXAMINATION

17.11.2022, 17:30

-SOLUTIONS-

Question	Grade	Out of
1		25
2		40
3		20
4		13
5		15
Total		113

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) SHOW ALL YOUR WORK. No points will be given to correct answers without reasonable work.

1. Find the solution sets of the following expressions. Clearly indicate the solution sets.

a) (9 points) $\boxed{\sqrt{x+3}+1=3\sqrt{x}}$ $\Rightarrow \left(\sqrt{x+3}+1\right)^2=\left(3\sqrt{x}\right)^2 \Rightarrow$

$\left((x+3)+2\sqrt{x+3}+1\right)=9x \Rightarrow \cancel{2}\sqrt{x+3}=8x-4=\cancel{2}(4x-2)$

$\Rightarrow \left(\sqrt{x+3}\right)^2=(4x-2)^2 \Rightarrow x+3=16x^2-16x+4 \Rightarrow \boxed{16x^2-17x+1=0}$

$\Rightarrow (16x-1)(x-1)=0 \Rightarrow \boxed{x_1=\frac{1}{16},\ x_2=1}$

Check in the original eqn. $\sqrt{x+3}+1=3\sqrt{x}$:

$x_1=\frac{1}{16} \Rightarrow \sqrt{\frac{1}{16}+3}+1 \overset{?}{=} 3\sqrt{\frac{1}{16}} \Rightarrow \frac{7}{4}+1=\frac{11}{4} \neq \frac{3}{4} \Rightarrow \left\{\frac{1}{16}\right\}$ is not a soln. of the original eqn.

$x_2=1 \Rightarrow \sqrt{1+3}+1 \overset{?}{=} 3\sqrt{1} \Rightarrow 3=3$ ✓

Soln. set: $\{1\}$

b) (8 points) $|3x-2|+x>\frac{5}{2}$

$|3x-2|>\frac{5}{2}-x \Rightarrow$

$3x-2>\frac{5}{2}-x \Rightarrow 4x>\frac{5}{2}+2=\frac{9}{2} \Rightarrow x>\frac{9}{8} \Rightarrow \left(\frac{9}{8},\infty\right)$

or

$-(3x-2)>\frac{5}{2}-x \Rightarrow -3x+x>\frac{5}{2}-2=\frac{1}{2} \Rightarrow -2x>\frac{1}{2} \Rightarrow x<-\frac{1}{4}$

$\Rightarrow \left(-\infty,-\frac{1}{4}\right)$

Soln. set: $\left(-\infty,-\frac{1}{4}\right) \cup \left(\frac{9}{8},\infty\right)$

c) (8 points) $|2x+5|+1 \geq 10$

$|2x+5| \geq 10-1=9 \Rightarrow 2x+5 \geq 9$ or $-(2x+5) \geq 9$

$2x \geq 9-5$

$2x \geq 4$

$[2,\infty) \Leftarrow x \geq 2$

$2x+5 \leq -9$

$\cancel{2}x \leq -9-5=\cancel{-14}\ -7$

$x \leq -7 \Rightarrow (-\infty,-7]$

Soln. set: $(-\infty,-7] \cup [2,\infty)$

2. a) (7 points) Write the following expression as a single logarithm in the most simplified form:

$\frac{1}{3}\ln x + 3\ln(x^2) - 2\ln(x-1) - 3\ln(x-2) = ?$

$\Rightarrow \ln x^{1/3} + \ln(x^2)^3 - \ln(x-1)^2 - \ln(x-2)^3$

$= \ln\left[\frac{x^{1/3}\cdot x^6}{(x-1)^2\cdot(x-2)^3}\right] = \boxed{\ln\left[\frac{x^{19/3}}{(x-1)^2\cdot(x-2)^3}\right]}$

b) (7 points) Simplify the expression: $\ln\sqrt[5]{\frac{(x+2)^2(x+9)^3}{(x+1)^4}}$.

$\ln\left(\frac{(x+2)^2\cdot(x+9)^3}{(x+1)^4}\right)^{1/5} = \ln(x+2)^{2/5} + \ln(x+9)^{3/5} - \ln(x+1)^{4/5}$

$= \boxed{\frac{2}{5}\ln(x+2) + \frac{3}{5}\ln(x+9) - \frac{4}{5}\ln(x+1)}$

c) (5 points) Find the solution set of: $3^{4x} = 9^{x+1}$.

$\boxed{3^{4x}} = 9^{x+1} = (3^2)^{x+1} = \boxed{3^{2x+2}} \Leftrightarrow 4x = 2x+2$

$\Rightarrow 2x = 2 \Rightarrow \boxed{x=1}$

$\boxed{\text{Soln. set: } \{1\}}$

d) (7 points) Find the solution set of: $\log(6x^2 - x - 11) = 0$.

$\hookrightarrow 10$

$\Rightarrow \underbrace{10^0}_{1} = 6x^2 - x - 11 \Rightarrow 6x^2 - x - 11 - 1 = 0 \Rightarrow 6x^2 - x - 12 = 0$

$\Rightarrow (3x+4)(2x-3) = 0$, $x = -\frac{4}{3}$, $x = \frac{3}{2}$ $\Rightarrow$ $\boxed{\text{Soln. set: } \{-\frac{4}{3}, \frac{3}{2}\}}$

e) (7 points) Find the solution set of: $\log_x(6 - 4x - x^2) = 2$.

$\hookrightarrow x > 0,\ x \neq 1$

$\Rightarrow x^2 = 6 - 4x - x^2 \Rightarrow 2x^2 + 4x - 6 = 0 \Rightarrow 2(x^2 + 2x - 3) = 0$

$\Rightarrow 2(x+3)(x-1) = 0$, $x = -3$ or $x = 1$

$x = -3$ contradicts with $x > 0$

$x = 1$ contradicts with $x \neq 1$

$\Rightarrow$ No solution $\Rightarrow$ $\boxed{\text{Soln. set: } \{\emptyset\}}$

f) (7 points) Find the solution set of: $\underbrace{\ln(x+1)}_{x>-1} + \underbrace{\ln(x+7)}_{x>-7} = \underbrace{\ln(2x-1)}_{x>\frac{1}{2}}$.

$-4 \quad -2$

$-7 \quad -1 \quad 1/2$

$\Rightarrow \ln[\underbrace{(x+1)(x+7)}_{x^2+8x+7}] = \ln(2x-1) \Rightarrow x^2 + 8x + 7 = 2x - 1$

$\Rightarrow x^2 + 6x + 8 = 0$

$(x+4)(x+2) = 0$

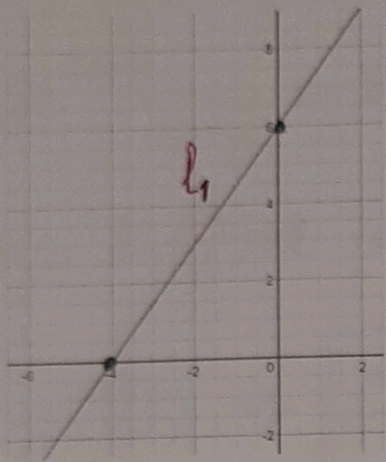
$\boxed{x = -4,\ x = -2}$

* $\ln(x+1)$ is defined for $x > -1$
* $\ln(x+7)$ " " " $x > -7$
* $\ln(2x-1)$ " " " $x > \frac{1}{2}$

all of them are defined when $x > \frac{1}{2}$

But $x = -4$ and $x = -2$ are less than $\frac{1}{2}$ $\Rightarrow$ there is no solution $\Rightarrow$ $\boxed{\text{Soln. set: } \{\emptyset\}}$

3. a) (6 points) Find the equation of the following line l_1 :



$$m_1 = \text{slope of } l_1 = \frac{6-0}{0-(-4)} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

$$y - 0 = \left(\frac{3}{2}\right)(x-(-4)) = \frac{3}{2}(x+4)$$

$$\boxed{y = \frac{3}{2}(x+4)} \Rightarrow \boxed{2y = 3x+12}$$

b) (7 points) Find the equation of the line l_2 passing through the point $(6, 2)$ and perpendicular to the line l_1 given in part a).

$$l_2 \perp l_1 \Rightarrow m_2 = \frac{-1}{m_1} = \frac{-1}{\frac{3}{2}} = \boxed{-\frac{2}{3}}$$

$$\Rightarrow y - 2 = \left(-\frac{2}{3}\right)(x-6) = -\frac{2}{3}x + 4$$

$$\Rightarrow \boxed{y = -\frac{2}{3}x + 6}$$

c) (7 points) Find the equation of the line l_3 passing through the point $(2, 2)$ and perpendicular to the line $\underbrace{3y - 6x + 9 = 0}$.

$$\text{line } l_4: \; 3y = 6x - 9 \Rightarrow y = 2x - 3$$

$$\downarrow$$

$$\text{slope } m_4 = 2$$

$$\Rightarrow l_3 \parallel l_4 \Rightarrow \underbrace{\text{slope } m_3 \text{ of } l_3}_{m_3 = 2} = \text{slope } m_4 \text{ of } l_4$$

$$l_3: \text{pt.}(2,2),\; m_3 = 2 \Rightarrow y - 2 = 2(x-2)$$

$$y = 2x - 4 + 2 \Rightarrow \boxed{y = 2x - 2}$$

4. a) (7 points) Find the inverse function $f^{-1}(x)$ given that $f(x) = 5e^{3x-2}$.

(Hint: Change the roles of x & y in $y = 5e^{3x-2}$ and use the fact that $y = e^u \iff \ln y = u$)

$$y = 5e^{3x-2} \xrightarrow[x \leftrightarrow y]{} x = 5e^{3y-2} \Rightarrow \frac{x}{5} = e^{3y-2} \Rightarrow \ln\left(\frac{x}{5}\right) = 3y-2$$

$$\Rightarrow 3y = \ln\left(\frac{x}{5}\right) + 2 \Rightarrow y = \frac{\ln\left(\frac{x}{5}\right)+2}{3}$$

$$\Rightarrow y = f^{-1}(x) \Rightarrow \boxed{f^{-1}(x) = \frac{\ln\left(\frac{x}{5}\right)+2}{3}}$$

b) (6 points) Find the inverse function $f^{-1}(x)$ given that $f(x) = \dfrac{x+2}{5x+4}$.

$$y = \frac{x+2}{5x+4} \xrightarrow[x \leftrightarrow y]{} x = \frac{y+2}{5y+4} \Rightarrow 5xy + 4x = y + 2$$

$$\Rightarrow y(5x-1) = 2-4x \Rightarrow y = \frac{2-4x}{5x-1}$$

$$\Rightarrow \boxed{y = f^{-1}(x) = -\frac{4x-2}{5x-1}}$$

5. Consider the function $f(x) = -(x+1)^2 + 8x + 1$.

a) (7 points) Find the vertex, x-intercept(s) and y-intercept(s) of $f(x)$ (If any).

$a=-1,\ b=6,\ c=0$

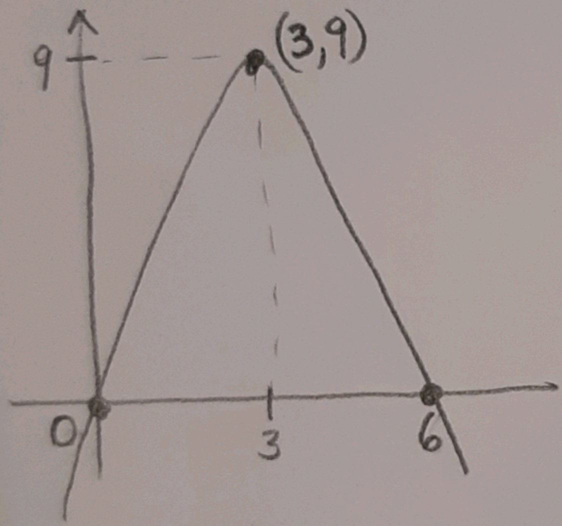
$$f(x) = -(x^2+2x+1)+8x+1 = -x^2+6x = -x(x-6)$$

Vertex: $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right) = \left(\underbrace{-\frac{6}{2(-1)}}_{3},\ f(3)\right) = \boxed{(3,9)}$

x-intercepts: $\boxed{(0,0),\ (6,0)}$

y-intercept: $\boxed{(0,0)}$

b) (4 points) Sketch the graph of f.



c) (4 points) Find the domain and the range of $f(x)$.

Domain $f(x)$: $\boxed{\mathbb{R}}$ (or $(-\infty, \infty)$)

Range $f(x)$: $\boxed{(-\infty, 9]}$