12.4: Implicit Differentiation:

* Some functions are defined implicitly in terms of their variables. For example,

$y = x^2 + 2x$ is an explicit function $\Rightarrow y' = 2x + 2$

$y^3 - 3yx + y^2x = \frac{1}{2}$ is an implicit function $\Rightarrow y' = ?$

* Implicit differentiation is a technique for differentiating functions that are not given in the usual form $y = f(x)$.

Implicit Differentiation Procedure:

Step 1: Differentiate both sides of the equation with respect to x.

Step 2: Collect all terms involving y' on one side of the equation.

Step 3: Factor y' from the side involving the y' terms

Step 4: Solve for y'.

Note: Remember Chain Rule properties:

1) $\frac{d}{dx}[y^n] = ny^{n-1} \cdot y'$

2) $\frac{d}{dx}[e^y] = e^y \cdot y'$ $\left(\frac{d}{dx}(a^y) = a^y \cdot y' \cdot \ln a\right)$

3) $\frac{d}{dx}(\ln y) = \frac{1}{y} \cdot y'$ $\left(\frac{d}{dx}(\log_a y) = \frac{1}{y} \cdot y' \cdot \frac{1}{\ln a}\right)$

Examples: Find y' for the followings:

1-) $x^2 + y^2 = 1 \Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \Rightarrow 2x + 2y \cdot y' = 0$

$\Rightarrow \cancel{2}y \cdot y' = -\cancel{2}x \Rightarrow \boxed{y' = -\frac{x}{y}}$

2-) $x^2y^3+y^2=x^4 \Rightarrow \frac{d}{dx}(x^2y^3+y^2)=\frac{d}{dx}(x^4)$

$\Rightarrow 2x\cdot y^3+x^2\cdot\left(3y^2\cdot\frac{dy}{dx}\right)+\left(2y\cdot\frac{dy}{dx}\right)=4x^3$

$\Rightarrow \frac{dy}{dx}(3x^2y^2+2y)=4x^3-2xy^3 \Rightarrow \boxed{\frac{dy}{dx}=\frac{4x^3-2xy^3}{3x^2y^2+2y}}$

3-) $y\cdot x+y^4\cdot x^4=y \Rightarrow \frac{d}{dx}(y\cdot x+y^4\cdot x^4)=\frac{d}{dx}(y)$

$\Rightarrow\left(\frac{dy}{dx}\cdot x+y(1)\right)+\left(\left(4y^3\cdot\frac{dy}{dx}\right)(x^4)+y^4(4x^3)\right)=\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left[x+4y^3\cdot x^4-1\right]=-y-4x^3y^4$

$\boxed{\frac{dy}{dx}=-\frac{y+4x^3y^4}{x+4y^3x^4-1}}$

4) $y\cdot\ln x+\frac{x}{y}=e^x \Rightarrow \frac{d}{dx}\left(y\cdot\ln x+\frac{x}{y}\right)=\frac{d}{dx}(e^x)$

$\Rightarrow\left(\frac{dy}{dx}\cdot\ln x+y\cdot\frac{1}{x}\right)+\left(1\cdot\frac{1}{y}+x\left(\frac{-1}{y^2}\cdot\frac{dy}{dx}\right)\right)=e^x$

$\frac{dy}{dx}\left[\ln x-\frac{x}{y^2}\right]=e^x-\frac{y}{x}-\frac{1}{y} \Rightarrow \boxed{\frac{dy}{dx}=\frac{e^x-\frac{y}{x}-\frac{1}{y}}{\ln x-\frac{x}{y^2}}}$

5-) $x^4\cdot e^y+y^5=\frac{1}{y} \Rightarrow \left(4x^3\cdot e^y+x^4\cdot e^y\frac{dy}{dx}\right)+\left(5y^4\cdot\frac{dy}{dx}\right)=\frac{-1}{y^2}\cdot\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left[x^4\cdot e^y+5y^4+\frac{1}{y^2}\right]=-4x^3\cdot e^y \Rightarrow \boxed{\frac{dy}{dx}=\frac{-4x^3\cdot e^y}{x^4\cdot e^y+5y^4+\frac{1}{y^2}}}$

6.) $y\ln x + x\ln y = x^2y^2 \Rightarrow \frac{d}{dx}\left(y\ln x + x\ln y\right) = \frac{d}{dx}\left(x^2y^2\right)$

$$\Rightarrow \left(\frac{dy}{dx}\cdot \ln x + y\cdot\frac{1}{x}\right) + \left(1\cdot \ln y + x\left(\frac{1}{y}\,\frac{dy}{dx}\right)\right) = 2x\cdot y^2 + x^2\left(2y\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left[\ln x + \frac{x}{y} - 2x^2y\right] = -\frac{y}{x} - \ln y + 2xy^2 = -\left[\frac{y}{x} + \ln y - 2xy^2\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\left(\frac{\frac{y}{x} + \ln y - 2xy^2}{\frac{x}{y} + \ln x - 2x^2y}\right)}$$

7.) $x^3 = (y-x^2)^2 \Rightarrow \frac{d}{dx}\left(x^3\right) = \frac{d}{dx}\left[\left(y-x^2\right)^2\right] \Rightarrow 3x^2 = 2(y-x^2)\cdot\left(\frac{dy}{dx} - 2x\right)$

$$\Rightarrow -2(y-x^2)\,\frac{dy}{dx} = -2x\left(2(y-x^2)\right) - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\left(2y-2x^2\right) + 3x^2}{2\left(y-x^2\right)} = \boxed{\frac{4xy - 4x^3 + 3x^2}{2\left(y-x^2\right)}}$$

8.) $\ln(xy) + x = 4 \Rightarrow \frac{d}{dx}\left(\ln(xy)\right) + \frac{d}{dx}(x) = \frac{d}{dx}(4)$

$$\Rightarrow \frac{1}{xy}\cdot\left[(1)(y) + (x)\left(\frac{dy}{dx}\right)\right] + 1 = 0 \Rightarrow \frac{\cancel{x}}{\cancel{x}y}\left(\frac{dy}{dx}\right) = -1 - \frac{\cancel{y}}{x\cancel{y}}$$

$$\Rightarrow \frac{1}{y}\left(\frac{dy}{dx}\right) = -\left(1 + \frac{1}{x}\right) \Rightarrow \boxed{\frac{dy}{dx}} = -\frac{1+\frac{1}{x}}{\frac{1}{y}} = \boxed{-\frac{y(x+1)}{x}}$$

9.) $\left(1+e^{3x}\right)^2 = 3 + \ln(x+y) \Rightarrow 2\left(1+e^{3x}\right)\cdot\left(3e^{3x}\right) = \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$

$$\Rightarrow \boxed{\left(6e^{3x}\right)\left(1+e^{3x}\right)(x+y) - 1 = \frac{dy}{dx}}$$

10.) $e^{x-y}=\ln(x-y) \Rightarrow \frac{d}{dx}\left(e^{x-y}\right)=\frac{d}{dx}\left(\ln(x-y)\right)$

$\Rightarrow e^{x-y}\cdot\left(1-\frac{dy}{dx}\right)=\frac{1}{x-y}\left(1-\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}\left[\frac{1}{x-y}-e^{x-y}\right]=\frac{1}{x-y}-e^{x-y} \Rightarrow \boxed{\frac{dy}{dx}=\frac{\cancel{\frac{1}{x-y}-e^{x-y}}}{\cancel{\frac{1}{x-y}-e^{x-y}}}}=\boxed{1}$

$\boxed{\frac{dy}{dx}=1}$

11.) $x^2+\ln y=5 \Rightarrow 2x+\frac{1}{y}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=y(-2x)$

$\Rightarrow \boxed{\frac{dy}{dx}=-2xy}$

12.) $e^x+e^y=x^2+y^2 \Rightarrow \frac{d}{dx}\left(e^x+e^y\right)=\frac{d}{dx}\left(x^2+y^2\right) \Rightarrow$

$\Rightarrow e^x+\left(e^y\cdot y'\right)=2x+\left(2y\cdot y'\right) \Rightarrow y'\left(e^y-2y\right)=2x-e^x \Rightarrow \boxed{y'=\frac{2x-e^x}{e^y-2y}}$

13.) $xe^y+y=13 \Rightarrow \frac{d}{dx}\left(xe^y+y\right)=\frac{d}{dx}(13) \Rightarrow 1.e^y+x.e^y\cdot\frac{dy}{dx}+\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}\left[xe^y+1\right]=-e^y \Rightarrow \boxed{\frac{dy}{dx}=\frac{-e^y}{xe^y+1}}$

14.) $y^2=e^{x+y} \Rightarrow \frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(e^{x+y}\right) \Rightarrow 2y\cdot y'=e^{x+y}\cdot\left(1+y'\right)$

$\Rightarrow y'\left(2y-e^{x+y}\right)=e^{x+y}$

$\Rightarrow \boxed{y'=\frac{e^{x+y}}{2y-e^{x+y}}}$

12.5: Logarithmic Differentiation:

A technique called logarithmic differentiation often simplifies the differentiation of $y=f(x)$ when $f(x)$ involves products, quotient or powers.

Logarithmic Differentiation Procedure: to differentiate $y=f(x)$;

- **Step 1:** Take the natural logarithm of both sides $\Rightarrow \ln(y)=\ln(f(x))$.
- **Step 2:** Simplify $\ln(f(x))$ by using properties of logarithms.
- **Step 3:** Differentiate both sides with respect to x.
- **Step 4:** Solve for y'.
- **Step 5:** Express the answer in terms of x only. This requires substituting $f(x)$ for y.

$\left(\begin{array}{l}\ln(A.B)=\ln A+\ln B\\ \ln A^n=n\ln A\end{array}\right)$

Examples: Find y':

1.) $y=(x+1)^2(x-2)(x^2+3)$: $\Rightarrow \ln y=\ln\left[(x+1)^2(x-2)(x^2+3)\right]$

$$=\ln(x+1)^2+\ln(x-2)+\ln(x^2+3)$$
$$=2\ln(x+1)+\ln(x-2)+\ln(x^2+3)$$

diff. w.r.t. x:

$$\Rightarrow \frac{y'}{y}=2\cdot\frac{1}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3} \Rightarrow y'=y\left[\frac{2}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3}\right]$$

$$\Rightarrow \boxed{y'=\left((x+1)^2(x-2)(x^2+3)\right)\left[\frac{2}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3}\right]}$$

2.) $y=4e^x.x^{3x} \Rightarrow \ln y=\ln\left(4e^x.x^{3x}\right)=\ln(4e^x)+\ln(x^{3x})$

$\Rightarrow \ln y=\ln(4e^x)+3x.(\ln x) \Rightarrow \frac{y'}{y}=\frac{\cancel{4e^x}}{\cancel{4e^x}}+\left(3.\ln x+3\cancel{x}.\frac{1}{\cancel{x}}\right)$

$\Rightarrow \frac{y'}{y}=1+3\ln x+3=3\ln x+4 \Rightarrow \boxed{y'=\underbrace{(4e^x.x^{3x})}_{y}.(3\ln x+4)}$

3.) $y=\dfrac{(1-3x)^2\cdot\sqrt{x^2+4}}{(x^3+7)^5} \Rightarrow \ln y=\ln\left(\dfrac{(1-3x)^2\cdot(x^2+4)^{1/2}}{(x^3+7)^5}\right)$

$= 2\ln(1-3x)+\frac{1}{2}\ln(x^2+4)-5\ln(x^3+7)$

$\Rightarrow \dfrac{y'}{y}=2\cdot\left(\dfrac{-3}{1-3x}\right)+\dfrac{1}{\cancel{2}}\left(\dfrac{\cancel{2}x}{x^2+4}\right)-5\left(\dfrac{3x^2}{x^3+7}\right)$

$\Rightarrow y'=y\left(\dfrac{-6}{1-3x}+\dfrac{x}{x^2+4}-\dfrac{15x^2}{x^3+7}\right)$

$\Rightarrow \boxed{y'=\left(\dfrac{(1-3x)^2\cdot(\sqrt{x^2+4})}{(x^3+7)^5}\right)\cdot\left(\dfrac{-6}{1-3x}+\dfrac{x}{x^2+4}-\dfrac{15x^2}{x^3+7}\right)}$

4.) $y=\left(\dfrac{3}{x^2}\right)^x \Rightarrow \ln y=\ln\left(\dfrac{3}{x^2}\right)^x = x\cdot(\ln 3-\ln x^2)$

$\Rightarrow \ln y = x(\ln 3-2\ln x)$

$\Rightarrow \dfrac{y'}{y}=(1)(\ln 3-2\ln x)+\underbrace{(x)\left(0-\dfrac{2}{x}\right)}_{-2}=\ln 3-2\ln x-2$

$\Rightarrow y'=y\cdot(\ln 3-2-2\ln x) = \boxed{\left(\dfrac{3}{x^2}\right)^x\cdot(\ln 3-2-2\ln x)}$

5.) $y=x^{x^2+1} \Rightarrow \ln y=\ln(x^{x^2+1})=(x^2+1)\cdot\ln x \quad (\ln A^r=r\ln A)$

$\Rightarrow \dfrac{y'}{y}=(2x)\cdot(\ln x)+(x^2+1)\left(\dfrac{1}{x}\right)=(2x)(\ln x)+x+\dfrac{1}{x}$

$\Rightarrow y'=y\left(2x(\ln x)+x+\dfrac{1}{x}\right)=\boxed{\left(x^{x^2+1}\right)\left(2x(\ln x)+x+\dfrac{1}{x}\right)}$

6-) $y=\sqrt[3]{\dfrac{6(x^3-1)^2}{x^6 \cdot e^{-4x}}} \Rightarrow \ln y = \ln\left(\dfrac{6(x^3-1)^2}{x^6 \cdot e^{-4x}}\right)^{1/3}$

$\Rightarrow \ln y = \frac{1}{3}\left[\ln 6 + \ln(x^3-1)^2 - \ln x^6 - \ln e^{-4x}\right]$ $\quad(\ln e^a = a)$

$= \frac{1}{3}\left[\ln 6 + 2\ln(x^3-1) - 6\ln x - (-4x)\right]$

$\Rightarrow \dfrac{y'}{y} = \dfrac{1}{3}\left[0 + 2\cdot\dfrac{3x^2}{x^3-1} - 6\cdot\dfrac{1}{x} + 4\right] = \left(\dfrac{2x^2}{x^3-1} - \dfrac{2}{x} + \dfrac{4}{3}\right)$

$\Rightarrow y' = y\cdot\left(\dfrac{2x^2}{x^3-1} - \dfrac{2}{x} + \dfrac{4}{3}\right) = \boxed{\left(\sqrt[3]{\dfrac{6(x^3-1)^2}{x^6 \cdot e^{-4x}}}\right)\left(\dfrac{2x^2}{x^3-1} - \dfrac{2}{x} + \dfrac{4}{3}\right)}$

7-) $y = (2x^2+1)\sqrt{8x^2-1} \Rightarrow \ln y = \ln\left[(2x^2+1)(8x^2-1)^{1/2}\right]$

$\Rightarrow \ln y = \ln(2x^2+1) + \frac{1}{2}\ln(8x^2-1)$

$\Rightarrow \dfrac{y'}{y} = \dfrac{4x}{2x^2+1} + \dfrac{1}{2}\,\dfrac{16x}{8x^2-1} \Rightarrow \boxed{y' = \left((2x^2+1)\sqrt{8x^2-1}\right)\left(\dfrac{4x}{2x^2+1} + \dfrac{8x}{8x^2-1}\right)}$

8-) $y = \dfrac{(2x-5)^3}{x^2 \cdot \sqrt[4]{x^2+1}} \Rightarrow \ln y = 3\ln(2x-5) - 2\ln(x) - \frac{1}{4}\ln(x^2+1)$

$\Rightarrow \dfrac{y'}{y} = 3\cdot\dfrac{2}{2x-5} - 2\,\dfrac{1}{x} - \dfrac{1}{4}\cdot\dfrac{2x}{x^2+1} = \dfrac{6}{2x-5} - \dfrac{2}{x} - \dfrac{x}{2(x^2+1)}$

$\Rightarrow \boxed{y' = \left(\dfrac{(2x-5)^3}{x^2 \cdot \sqrt[4]{x^2+1}}\right)\cdot\left(\dfrac{6}{2x-5} - \dfrac{2}{x} - \dfrac{x}{2(x^2+1)}\right)}$

12.7: Higher Order Derivatives:

Derivative of a function $y=f(x)$ is itself a function of x; $f'(x)$. If we differentiate $f'(x)$, the resulting function is called the second derivative of f at x. It is denoted by $f''(x)$. Third and higher order derivatives are defined similarly.

Notations: First Der.: y', $\frac{dy}{dx}$, $f'(x)$, $\frac{df}{dx}$

Second Der.: y'', $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d^2f}{dx^2}$

Third Der.: y''', $\frac{d^3y}{dx^3}$, $f'''(x)$, $\frac{d^3f}{dx^3}$

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Examples:

1-) Let $f = 6x^3 - 12x^2 + 6x - 2$. Find all derivatives of f.

$f' = 18x^2 - 24x + 6 \Rightarrow f'' = 36x - 24 \Rightarrow f''' = 36 \Rightarrow f^{(IV)}(x) = 0$

$\Rightarrow$ all order derivatives greater or equal to order 4 are zero.

$\Rightarrow$ $\boxed{\deg f = 3 \Rightarrow \text{derivatives of } f \text{ of order greater than } \deg f = \text{zero}}$

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2) $y = x.e^{x^3} \Rightarrow$ Find $y'' = ?$

$y' = (1)(e^{x^3}) + (x).(e^{x^3}.3x^2) = e^{x^3}(1+3x^3)$ (Product)

$y'' = (e^{x^3}.3x^2)(1+3x^3) + (e^{x^3}).(9x^2)$

$= 3x^2.e^{x^3} + 9x^5.e^{x^3} + 9x^2.e^{x^3} = \boxed{(12x^2 + 9x^5)e^{x^3}}$

3.) Let $y=\frac{16}{x+4}$. Find $y''(4)$.

$$y'=\frac{(0)\cancel{(x+4)}-(1)(16)}{(x+4)^2}=\frac{-16}{(x+4)^2}\Rightarrow y''=\frac{(0)\cancel{(x+4)^2}-(-16)(2(x+4))}{\left((x+4)^2\right)^2}$$

$$\Rightarrow y''=\frac{32(x+4)}{(x+4)^4}=\frac{32}{(x+4)^3}\Rightarrow y''(4)=\frac{32}{(4+4)^3}=\frac{32}{8^3}=\frac{4}{8^2}=\frac{1}{16}$$

4.) Let $y=\ln\left[\frac{(2x+5)(5x-2)}{(x+1)}\right]$. Find $y''=?$

$$\Rightarrow \quad y=\ln(2x+5)+\ln(5x-2)-\ln(x+1)$$

$$y'=\frac{2}{2x+5}+\frac{5}{5x-2}-\frac{1}{x+1}$$

$$y''=\frac{(0)\cancel{(2x+5)}-(2)(2)}{(2x+5)^2}+\frac{(0)\cancel{(5x-2)}-(5)(5)}{(5x-2)^2}-\frac{(0)\cancel{(x+1)}-(1)(1)}{(x+1)^2}$$

$$\boxed{y''=\frac{-4}{(2x+5)^2}-\frac{25}{(5x-2)^2}+\frac{1}{(x+1)^2}}$$

5.) $y=\frac{x}{e^x}\Rightarrow y''(0)=?$

$$y'=\frac{(1)(e^x)-(e^x)(x)}{(e^x)^2}=\frac{\cancel{e^x}(1-x)}{e^{2x}}=\frac{1-x}{e^x}$$

$$y''=\frac{(-1)(e^x)-(e^x)(1-x)}{(e^x)^2}=\frac{\cancel{e^x}(-2+x)}{e^{2x}}=\boxed{\frac{x-2}{e^x}}$$

$$y''(0)=\frac{0-2}{e^0}=\frac{-2}{1}=\boxed{-2}$$