ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 113 - Mathematics for City Planners

2021-2022 Fall

FINAL EXAM

10.01.2022, 10:00

SOLUTIONS

Question	Grade	Out of
1		14
2		14
3		14
4		12
5		20
6		18
7		21
Total		113

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 7 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Solve the following equations for **x**. Clearly explain and indicate your answers.

(7 pts.) a) $\boxed{\log(5x+5) = \log(3x-2)+1}$

$\log(5x+5) - \log(3x-2) = 1 \Rightarrow \log\left(\frac{5x+5}{3x-2}\right) = 1$

$\Rightarrow 10^1 = \frac{5x+5}{3x-2} \Rightarrow (3x-2)(10) = 5x+5 \Rightarrow 25x = 25 \Rightarrow \boxed{x=1}$

$\log(5x+5)$ is defined when $5x+5>0 \Rightarrow x>-1$

$\log(3x-2)$ is " " $3x-2>0 \Rightarrow x>\frac{2}{3}$

$x=1$ satisfies both these conditions $\Rightarrow$

So, soln.: $\Rightarrow \boxed{x=1}$

(7 pts.) b) $\boxed{e^{2\ln(2x)} = 4}$

$e^{2\ln(2x)} = e^{\ln(2x)^2} = (2x)^2 = 4x^2 = 4$

$\Rightarrow 4x^2 = 4 \Rightarrow x^2 = 1 \Rightarrow \boxed{x = \mp 1}$

But $\ln(2x)$ is defined for $2x>0 \Rightarrow x>0$

So $x=-1$ cannot be a soln. $\Rightarrow$ So; soln: $\boxed{x=1}$

2. Find the domains of each of the following functions:
(Indicate your answer as an interval (or union of intervals)).

(7 pts.) a) $\boxed{f(x) = \frac{x}{\sqrt{x^2-9}} + \log_5(x-3)}$

$\frac{x}{\sqrt{x^2-9}}$ is defined for x's s.t.: $x^2-9>0 \Rightarrow \boxed{x<-3 \text{ or } x>3}$ (A)

$\log_5(x-3)$ is defined for $x-3>0 \Rightarrow \boxed{x>3}$ (B)

Combining (A) & (B): domain $f(x)$: $\{x \mid x>3\} \Rightarrow \boxed{(3,\infty)}$

(7 pts.) b) $\boxed{g(x) = e^{\left(\frac{x^2-x-12}{x^2+5x-6}\right)}}$ is defined when $\frac{x^2-x-12}{x^2+5x-6}$ is defined $\Rightarrow$

$\frac{x^2-x-12}{x^2+5x-6} = \frac{(x-4)(x+3)}{(x+6)(x-1)}$ is defined when $x \neq -6$ and $x \neq 1 \Rightarrow$

$\Rightarrow$ So domain $g(x) = \boxed{(-\infty,-6)\cup(-6,1)\cup(1,\infty)}$ (or $\mathbb{R}\setminus\{-6,1\}$)

3. Evaluate the following limits. Show your work and do not use L'Hopital's rule!

(7 pts.) **a)** $\lim_{x\to\infty}\left[\frac{x^2+x+1}{(3x+2)^2}\right]$ $\left((3x+2)^2 = 9x^2+12x+4\right)$

$$\lim_{x\to\infty}\left[\frac{x^2+x+1}{9x^2+12x+4}\right] = \lim_{x\to\infty}\left[\frac{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(9+\frac{12}{x}+\frac{4}{x^2}\right)}\right]$$

$$= \frac{1+0+0}{9+0+0} = \boxed{\frac{1}{9}}$$

(7 pts.) **b)** $\lim_{x\to-\infty}\left(e^{-x}+2x^2+5\right)$

$$\lim_{x\to-\infty}\left(e^{-x}+2x^2+5\right) = e^{-(-\infty)}+2(-\infty)^2+5$$

$$= e^{\infty}+2(\infty)+5$$

$$= \infty+\infty+5 = \boxed{\infty}$$

(12 pts.) **4.** Let $f(x) = \begin{cases} x^2+3x+5 & \text{if } x<-2 \\ 3 & \text{if } x=-2 \\ x^3+2x^2-5x-7 & \text{if } x>-2 \end{cases}$. Find $f'(-2)$ if it exists.

$\left(\text{Hint}: f'(-2) = \lim_{x\to-2}\left[\frac{f(x)-f(-2)}{x-(-2)}\right]\right)$

$\boxed{\text{left der. of } f \text{ at } x=-2}$ $\Rightarrow \lim_{\substack{x\to-2^- \\ (x<-2)}}\left[\frac{f(x)-f(-2)}{x-(-2)}\right] = \lim_{x\to-2^-}\left[\frac{(x^2+3x+5)-3}{x+2}\right] = \lim_{\substack{x\to-2^- \\ (x\neq-2)}}\left[\frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}}\right]$

$$= (-2+1) = -1$$

$\boxed{\text{right der. of } f \text{ at } x=-2}$ $\Rightarrow \lim_{\substack{x\to-2^+ \\ (x>-2)}}\left[\frac{f(x)-f(-2)}{x-(-2)}\right] = \lim_{x\to-2^+}\left[\frac{\overbrace{(x^3+2x^2-5x-7)-3}^{x^3+2x^2-5x-10}}{x+2}\right] =$

$$= \lim_{\substack{x\to-2^+ \\ (x\neq-2)}}\left[\frac{(x^2-5)\cancel{(x+2)}}{\cancel{(x+2)}}\right] = (-2)^2-5 = \boxed{-1}$$

since right & left derivatives at $x=-2$ are equal $\Rightarrow$

$$\boxed{f'(-2) = -1}$$

5. (10 pts.) a) Write the equation of the tangent line to the curve $x^8 + 4x^2y^2 + y^8 = 6$ at the point $(1, 1)$.

$$8x^7 + 4(2x \cdot y^2 + x^2 \cdot 2y \cdot y') + 8y^7 \cdot y' = 0$$

$$\Rightarrow \boxed{y'} = -\frac{8x^7 + 8xy^2}{8y^7 + 8x^2y} = -\frac{\cancel{8}(x^7 + xy^2)}{\cancel{8}(y^7 + x^2y)} = \boxed{-\frac{x^7 + xy^2}{y^7 + x^2y}}$$

$$\Rightarrow y'\Big|_{(1,1)} = -\frac{1+1}{1+1} = \boxed{-1} = m = \text{slope of tangent line}$$

$$y - 1 = (-1)(x-1) \Rightarrow y = -x + 1 + 1 \Rightarrow \boxed{y = -x + 2}$$ eqn. of tg. line.

$$(y - y_0 = m(x - x_0))$$

(10 pts.) b) Let $y = \ln\left[\frac{(2x+5)(5x-2)}{(x+1)}\right]$. Find $y''(0) = ?$

$$y = \ln(2x+5) + \ln(5x-2) - \ln(x+1)$$

$$y' = \frac{2}{2x+5} + \frac{5}{5x-2} - \frac{1}{x+1}$$

$$y'' = \frac{-4}{(2x+5)^2} - \frac{25}{(5x-2)^2} + \frac{1}{(x+1)^2}$$

$$y''(0) = \frac{-4}{(0+5)^2} - \frac{25}{(0-2)^2} + \frac{1}{(0+1)^2} = \underset{(4)}{\frac{-4}{25}} - \underset{(25)}{\frac{25}{4}} + \underset{(100)}{\frac{1}{1}}$$

$$= \frac{-16 - 625 + 100}{100} = \boxed{\frac{541}{100}}$$

6. Find y' in the following expressions. Simplify your answer as much as possible.

(9 pts.) **a)** $y = \sqrt{1+\sqrt{1+x}}$

$$y=\left(1+(1+x)^{1/2}\right)^{1/2} \Rightarrow y'=\frac{1}{2}\left(1+(1+x)^{1/2}\right)^{-1/2}\cdot\left(\frac{1}{2}(1+x)^{-1/2}\right)$$

$$\Rightarrow \boxed{y'=\frac{1}{4}\cdot\frac{1}{\sqrt{1+\sqrt{1+x}}}\cdot\frac{1}{\sqrt{1+x}}}$$

(9 pts.) **b)** $y = 5^{x+5\log_5 x}$ $= 5^x \cdot 5^{5\log_5 x}$

$$\boxed{y} = 5^x\cdot \underbrace{\underbrace{5^{5\log_5 x}}_{5^{\log_5 x^5}}}_{x^5} = \boxed{5^x\cdot x^5}$$

$$y'=(5^x\cdot \ln 5)(x^5)+(5^x)(5x^4) \Rightarrow$$

$$\boxed{y'=5^x\cdot x^4\,(x\cdot \ln 5+5)}$$

7. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 3 & 4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 3 \\ 5 & 8 \\ 2 & 4 \end{bmatrix}$$

If it is possible, compute the followings. If it is not possible, explain why.

(3 pts.) a) $2A + B$

A: 2x4 matrix $\Rightarrow$ 2A: 2x4 matrix

B: 2x2 matrix

$\boxed{\text{2A+B: not possible}}$ since the sizes of 2A & B are not the same.

(3 pts.) b) $C^T B$

$$C^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2} \cdot B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}_{2\times 2} = \boxed{\begin{bmatrix} 5 & 6 \\ 11 & 14 \\ 15 & 18 \end{bmatrix}}_{3\times 2}$$

(3 pts.) c) $(CD)^T$

$$(CD)^T = D^T \cdot C^T = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 8 & 4 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2} = \boxed{\begin{bmatrix} 15 & 5 \\ 31 & 8 \end{bmatrix}}_{2\times 2}$$

(3 pts.) d) CB

$C_{2\times 3} \cdot B_{2\times 2}$: $\boxed{\text{not possible}}$ since $\boxed{\text{\# of columns of C} \neq \text{\# of rows of B}}$

$3 \neq 2$

(3 pts.) e) $CD + B$

$$CD = ((CD)^T)^T = \begin{bmatrix} 15 & 5 \\ 31 & 8 \end{bmatrix}^T = \begin{bmatrix} 15 & 31 \\ 5 & 8 \end{bmatrix} + B = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 20 & 37 \\ 6 & 10 \end{bmatrix}}$$

by part (c)

(3 pts.) f) trace B

$$\text{trace} \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = 5 + 2 = \boxed{7}$$

(3 pts.) g) trace C

trace C: $\boxed{\text{not possible}}$ since C is not a square matrix.