11.5: CHAIN RULE

If f is a differentiable function of u, and u is a differentiable function of x, the f is a differentiable function of x and;

$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}$$

$$\boxed{f \downarrow u \downarrow x}$$

Rules of differentiation:

1. $\left([f(x)]^n\right)' = n\left(f(x)\right)^{n-1} \cdot f'(x)$
2. $\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x) \quad \left[\left(b^{f(x)}\right)' = b^{f(x)} \cdot f'(x) \cdot \ln b\right]$
3. $\left(\ln f(x)\right)' = \frac{1}{f(x)} \cdot f'(x) \quad \left[\left(\log_b f(x)\right)' = \frac{1}{f(x) \cdot \ln b} \cdot f'(x)\right]$

Examples: Evaluate the derivative of the following functions:

1) $f(x) = (x^2+9)^2 \Rightarrow f'(x) = 2(x^2+9)\cdot(2x) = 4x(x^2+9) = \boxed{4x^3+36x}$

2) $f(x) = (x^3+x+5)^{21} \Rightarrow f'(0) = ?$

$f'(x) = 21(x^3+x+5)^{20}\cdot(3x^2+1) \Rightarrow f'(0) = 21(5)^{20}(1) = \boxed{21 \cdot 5^{20}}$

3) $f(x) = (3x^2-1)^7 \Rightarrow f'(1) = ?$

$f'(x) = 7(3x^2-1)^6 \cdot (6x) \Rightarrow f'(1) = 7(3-1)^6(6(1)) = 7 \cdot 2^6 \cdot 6 = \boxed{42 \cdot 64}$

4) $f(x) = (x^4-4x^3)^2 \Rightarrow f'(x) = \boxed{2(x^4-4x^3)\cdot(4x^3-12x^2)}$

5) $f(x) = \sqrt{1-x^2} = (1-x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-\cancel{2}x) = \boxed{\frac{-x}{\sqrt{1-x^2}}}$

6.) $f(x) = \frac{1}{\sqrt{x - x^5}} = (x - x^5)^{-1/2}$

$f'(x) = -\frac{1}{2}(x - x^5)^{-3/2} \cdot (1 - 5x^4) = \boxed{-\frac{(1-5x^4)}{2\sqrt{(x-x^5)^3}}}$

7.) $f(x) = \sqrt{(x^3+1)(4x-5)} = \left[(x^3+1)(4x-5)\right]^{1/2}$

$f'(x) = \frac{1}{2}\left[(x^3+1)(4x-5)\right]^{-1/2} \cdot \underbrace{\left((3x^2)(4x-5) + (x^3+1)(4)\right)}_{12x^3 - 15x^2 + 4x^3 + 4}$

$= \boxed{\frac{(16x^3 - 15x^2 + 4)}{2\sqrt{(x^3+1)(4x-5)}}}$

8.) $f(x) = \left(\frac{6x+4}{3-x^2}\right)^{7/3} \Rightarrow f'(x) = \frac{7}{3}\left(\frac{6x+4}{3-x^2}\right)^{7/3 - 1} \cdot \left(\frac{6x+4}{3-x^2}\right)'$

$\Rightarrow f'(x) = \frac{7}{3}\left(\frac{6x+4}{3-x^2}\right)^{\frac{4}{3}} \cdot \left(\frac{(6)(3-x^2) - (-2x)(6x+4)}{(3-x^2)^2}\right) = \boxed{\frac{7}{3}\left(\frac{6x+4}{3-x^2}\right)^{4/3} \cdot \left(\frac{6x^2+8x+18}{(3-x^2)^2}\right)}$

9.) $f(x) = e^{x^2+3x} \Rightarrow f'(x) = \boxed{e^{x^2+3x} \cdot (2x+3)}$

10.) $f(x) = \ln(x^2+3x) \Rightarrow f'(x) = \boxed{\frac{(2x+3)}{x^2+3x}}$

11.) $f(x) = 2^{\sqrt{x^2+5x}} = 2^{(x^2+5x)^{1/2}} \Rightarrow f'(x) = 2^{\sqrt{x^2+5x}} \cdot \frac{1}{2}(x^2+5x)^{-1/2} \cdot (2x+5) . \ln 2$

$\Rightarrow f'(x) = \boxed{2^{\sqrt{x^2+5x}} \cdot \frac{(2x+5)(\ln 2)}{2\sqrt{x^2+5x}}}$

12.) $f(x) = \log_7(x^3 - 7x^2)$

$\left(* \ \log_b a = \frac{\ln a}{\ln b}\right)$

$\boxed{f'(x) = \frac{1}{x^3 - 7x^2} \cdot (3x^2 - 14x) \cdot \frac{1}{\ln 7}}$

13.) $f(x) = e^{x^3+\sqrt{x}+2} \Rightarrow f'(x) = \boxed{e^{x^3+\sqrt{x}+2} \cdot \left(3x^2 + \frac{1}{2\sqrt{x}}\right)}$

14.) $f(x) = \ln(x^4+3x^2+8) \Rightarrow f'(x) = \boxed{\frac{4x^3+6x}{x^4+3x^2+8}}$

15.) $f(x) = \ln\left(x^{3/2}+x+1\right) \Rightarrow f'(x) = \boxed{\frac{\frac{3}{2}x^{1/2}+1}{x^{3/2}+x+1}}$

16.) $f(x) = \sqrt{x+\ln x} = (x+\ln x)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x+\ln x)^{-1/2} \cdot \left(1+\frac{1}{x}\right)$

$\Rightarrow f'(x) = \boxed{\frac{1}{2\sqrt{x+\ln x}} \cdot \left(1+\frac{1}{x}\right)}$

17.) $f(x) = (1+x+\ln x+2^x)^7 \Rightarrow f'(x) = \boxed{7(1+x+\ln x+2^x)^6 \cdot \left(1+\frac{1}{x}+2^x \cdot \ln 2\right)}$

18.) $f(x) = (x^2+\ln(x^2+4))^2 \Rightarrow f'(x) = \boxed{2(x^2+\ln(x^2+4)) \cdot \left(2x+\frac{2x}{x^2+4}\right)}$

19.) $f(x) = \left(x^2+4x+e^{x^3+x}\right)^5$

$f'(x) = \boxed{5\left(x^2+4x+e^{x^3+x}\right)^4 \cdot \left(2x+4+e^{x^3+x} \cdot (3x^2+1)\right)}$

20.) $f(x) = \ln\left[(1+x^2)(x^3-7x)\right] = \ln(1+x^2) + \ln(x^3-7x)$

$\ln(A \cdot B) = \ln A + \ln B$

$\Rightarrow f'(x) = \boxed{\frac{2x}{1+x^2} + \frac{3x^2-7}{x^3-7x}}$

21.) $f(x) = \ln\left(\frac{x-2}{3x+5}\right) = \ln(x-2) - \ln(3x+5)$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$f'(x) = \boxed{\frac{1}{x-2} - \frac{3}{3x+5}}$

22-) $f(x) = \ln(1+4x^3)^8 = 8\ln(1+4x^3)$

$\ln A^r = r\ln A$

$$\Rightarrow f'(x) = 8\left(\frac{12x^2}{1+4x^3}\right) = \boxed{\frac{96x^2}{1+4x^3}}$$

23-) $f(x) = \ln(\ln x) \Rightarrow f'(x) = \boxed{\frac{1}{\ln x}\cdot\frac{1}{x}}$

24-) $f(x) = e^{(3x^4+1)^2} \Rightarrow f'(x) = \boxed{e^{(3x^4+1)^2}\cdot 2(3x^4+1)\cdot(12x^3)}$

25-) $f(x) = \dfrac{e^{2x}}{1-e^{3x}} \Rightarrow f'(x) = \dfrac{(e^{2x}\cdot 2)(1-e^{3x}) - (-e^{3x}\cdot 3)(e^{2x})}{(1-e^{3x})^2}$

$$\Rightarrow f'(x) = \frac{2e^{2x} - 2e^{5x} + 3e^{5x}}{(1-e^{3x})^2} = \boxed{\frac{e^{5x} - 2e^{2x}}{(1-e^{3x})^2}}$$

26-) $f(x) = \sqrt{1-e^{-x}}$

$$f(x) = (1-e^{-x})^{1/2} \Rightarrow f'(x) = \frac{1}{2}(1-e^{-x})^{-1/2}\cdot(-e^{-x}\cdot(-1)) = \boxed{\frac{e^{-x}}{2\sqrt{1-e^{-x}}}}$$

27-) $f(x) = \ln(x + x^2\cdot e^{2x}) \Rightarrow f'(x) = \dfrac{1}{x+x^2\cdot e^{2x}}(1 + 2xe^{2x} + x^2\cdot e^{2x}\cdot 2)$

$$\Rightarrow f'(x) = \boxed{\frac{1+2xe^{2x}+2x^2e^{2x}}{x+x^2e^{2x}}}$$

28-) $f(x) = \ln(1-e^{x^2+2x}) \Rightarrow f'(x) = \dfrac{1}{1-e^{x^2+2x}}\cdot\left(0 - e^{x^2+2x}\cdot(2x+2)\right)$

$$\Rightarrow f'(x) = \boxed{\frac{-(2x+2)e^{x^2+2x}}{1-e^{x^2+2x}}}$$

29-) $f(x) = e^{x^2+2\ln x} \Rightarrow f'(x) = \underbrace{e^{x^2+2\ln x}}_{x^2\cdot e^{x^2}}\cdot\left(2x + \dfrac{2}{x}\right)$

$= \boxed{e^{x^2}(2x^3+2x)}$

(<u>Also</u>; $f(x) = e^{x^2}\cdot e^{2\ln x} = e^{x^2}\cdot \underbrace{e^{\ln x^2}}_{x^2}$

$= \boxed{x^2\cdot e^{x^2}}$)

30) $f(x)=e^{x^3+3x}\cdot\ln(x^2+1)$

$$f'(x)=\left[e^{x^3+3x}\cdot(3x^2+3)\right]\left[\ln(x^2+1)\right]+\left[\left(\frac{2x}{x^2+1}\right)\right]\left[e^{x^3+3x}\right]$$

$$=\boxed{e^{x^3+3x}\cdot\left[(3x^2+3)\ln(x^2+1)+\frac{2x}{x^2+1}\right]}$$

---

31) Given that $f(0)=2$, $g(0)=3$, $f'(0)=5$, $g'(0)=7$, $f'(3)=\pi$ and $g'(2)=\pi^2$;

a) If $h(x)=f(x)\cdot g(x)$, compute $h'(0)$?

$$h'(x)=f'(x)\cdot g(x)+g'(x)\cdot f(x)\Rightarrow h'(0)=f'(0)\cdot g(0)+g'(0)\cdot f(0)$$
$$=(5)(3)+(7)(2)=15+14=29$$

---

b) If $k(x)=\dfrac{f(x)}{g(x)}$, compute $k'(0)$?

$$k'(x)=\frac{f'(x)\cdot g(x)-g'(x)\cdot f(x)}{(g(x))^2}\Rightarrow k'(0)=\frac{f'(0)\cdot g(0)-g'(0)\cdot f(0)}{(g(0))^2}$$

$$\Rightarrow k'(0)=\frac{(5)(3)-(7)(2)}{(3)^2}=\frac{15-14}{9}=\frac{1}{9}$$

---

c) If $s(x)=f(g(x))$ and $t(x)=g(f(x))$, compute $s'(0)$ and $t'(0)$?

$$s'(x)=f'(g(x))\cdot g'(x)\Rightarrow s'(0)=f'(\underbrace{g(0)}_{3})\cdot g'(0)=\underbrace{f'(3)}_{\pi}\cdot 7=7\pi$$

$$t'(x)=g'(f(x))\cdot f'(x)\Rightarrow t'(0)=g'(\underbrace{f(0)}_{2})\cdot f'(0)=\underbrace{g'(2)}_{\pi^2}\cdot 5=5\pi^2$$

---

32) Let $k(x)=f(g(h(x)))$ where $h(1)=2$, $g(2)=3$, $h'(1)=4$, $g'(2)=5$, $f'(3)=6$.
Find $k'(1)$?

$$k'(x)=f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)\Rightarrow k'(1)=f'(g(\underbrace{h(1)}_{2}))\cdot g'(\underbrace{h(1)}_{2})\cdot \underbrace{h'(1)}_{4}$$

$$\Rightarrow k'(1)=f'(\underbrace{g(2)}_{3})\cdot \underbrace{g'(2)}_{5}\cdot 4=\underbrace{f'(3)}_{6}\cdot 5\cdot 4=6\cdot 5\cdot 4=\boxed{120}$$

33.) Suppose f and g are differentiable functions with the values for the function and derivative given in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	7	9	5
3	1	9	7	9
5	9	5	5	3
7	7	1	3	1
9	5	3	1	7

a.) If $\boxed{h(x)=f(x)+2g(x)}$, find $\boxed{h'(1)}$:

$$h'(x)=f'(x)+2g'(x) \Rightarrow h'(1)=f'(1)+2g'(1)=7+2(5)=17$$

b.) If $\boxed{h(x)=\frac{x\cdot f(x)}{g(x)}}$, find $\boxed{h'(5)}$:

$$h'(x)=\frac{[1.f(x)+x.f'(x)][g(x)]-[g'(x)][x.f(x)]}{[g(x)]^2}$$

$$h'(5)=\frac{(f(5)+5.f'(5))(g(5))-(g'(5))(5.f(5))}{(g(5))^2}=\frac{(9+5.5)(5)-(3)(5.9)}{(5)^2}$$

$$=\frac{170-135}{25}=\frac{35}{25}=\frac{7}{5}$$

c.) If $\boxed{h(x)=f(g(x^2))}$, find $\boxed{h'(3)}$:

$$h'(x)=f'(g(x^2)).g'(x^2).2x \Rightarrow h'(3)=f'(g(3^2)).g'(3^2).(2.3)$$

$$\Rightarrow h'(3)=f'(\underbrace{g(9)}_{1}).\underbrace{g'(9)}_{7}.6=\underbrace{f'(1)}_{7}.7.6=7.7.6=\boxed{294}$$