Math 113 : 13/10/2020

Rules of inequalities:

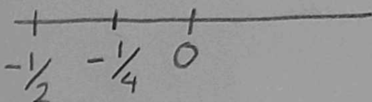
1.) If $a<b$ then $a+c<b+c$ and $a-c<b-c$

2.) If $a<b$ and $c>0 \Rightarrow ac<bc$, $\frac{a}{c}<\frac{b}{c}$

3.) If $a<b$ and $c<0 \Rightarrow ac>bc$, $\frac{a}{c}>\frac{b}{c}$

4.) If $0<a<b \Rightarrow \frac{1}{b}<\frac{1}{a}$ $\left(0<2<4,\ \frac{1}{4}<\frac{1}{2}\right)$

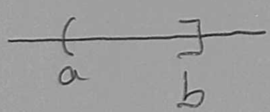
If $a<b<0 \Rightarrow \frac{1}{b}<\frac{1}{a}$ $\left(\text{eg. } \underset{a}{-4}<\underset{b}{-2}<0 \Rightarrow \overset{\frac{1}{b}}{-\frac{1}{2}}<\overset{\frac{1}{a}}{-\frac{1}{4}}\right)$

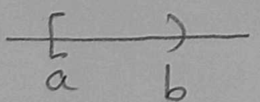


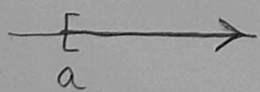
5.) If $0<a<b$, $n>0 \Rightarrow a^n<b^n$ $(0<2<3 \Rightarrow 2^2<3^2)$

If $0<a<b$, $n>0 \Rightarrow \sqrt[n]{a}<\sqrt[n]{b}$

Intervals:

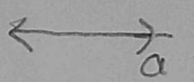
- $(a,b]$:  , $a<x\leq b$

- $[a,b)$:  , $a\leq x<b$

- $[a,\infty)$:  , $a\leq x<\infty$

- (a,∞) :  , $a<x<\infty$

- $(-\infty,a]$:  , $-\infty<x\leq a$

- $(-\infty,a)$:  , $-\infty<x<a$

- $(-\infty,\infty)$:  , $-\infty<x<\infty$

ex: Solve the following inequalities. Express the solution as an interval or union of intervals.

a.) $3x-2 < x+18 \Rightarrow 3x-x < 18+2 \Rightarrow 2x < 20 \Rightarrow x < 10$

$\boxed{\text{Soln. set: } (-\infty, 10)}$

b.) $\frac{3x-1}{4} \geq \frac{1-x}{2} \Rightarrow 2(3x-1) \geq 4(1-x) \Rightarrow 6x-2 \geq 4-4x$

$\Rightarrow 6x+4x \geq 4+2 \Rightarrow 10x \geq 6 \Rightarrow x \geq \frac{6}{10}$

$\Rightarrow x \geq \frac{3}{5} \Rightarrow \boxed{\text{Soln. set: } \left[\frac{3}{5}, \infty\right)}$

c.) $-3x-1 < 3+x \Rightarrow -1-3 < x+3x \Rightarrow -4 < 4x \Rightarrow \frac{-4}{4} < \frac{4x}{4}$

$\Rightarrow -1 < x \Rightarrow \boxed{(-1, \infty)}$

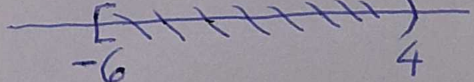
d.) $2(4x-2) > 4(2x+1) \Rightarrow \underset{-8x}{\cancel{8x}-4} > \underset{-8x}{\cancel{8x}+4} \Rightarrow -4 > 4$

imp.

$\Rightarrow$ Soln. set: $\{\emptyset\}$ or There is no soln.

e.) $\underset{-2}{-6} < \underset{-2}{2-2x} \leq \underset{-2}{14} \Rightarrow -6-2 < \cancel{2}-2x-\cancel{2} \leq 14-2 \Rightarrow \boxed{-8 < -2x \leq 12}$

$\frac{-8}{-2} > \frac{-2x}{-2} \geq \frac{12}{-2} \Rightarrow 4 > x \geq -6 \Rightarrow$



Soln. set: $\boxed{[-6, 4)}$

f_1-) $x^2 \leq 2x+3 \Rightarrow x^2-2x-3 \leq 0 \Rightarrow (x-3)(x+1) \leq 0$

$x-3=0 \Rightarrow x=3$

$x+1=0 \Rightarrow x=-1$

	$-\infty$	-1		3	∞
$x+1$	$-$		$+$		$-$
$x-3$	$-$		$-$		$+$
$(x+1)(x-3)$	$+$		$-$		$+$

$\boxed{\text{Soln. set: } [-1, 3]}$

f_2-) $x^2 \geq 2x+3 \Rightarrow (x-3)(x+1) \geq 0$

Soln. set: $\boxed{(-\infty, -1] \cup [3, \infty)}$

g_1-) $\dfrac{x-1}{3-x} > 2 \Rightarrow \dfrac{x-1}{3-x} - 2 > 0 \Rightarrow \boxed{\dfrac{3x-7}{3-x} > 0}$

	$-\infty$	$7/3$		3	∞
$3x-7$	$-$		$+$		$+$
$3-x$	$+$		$+$		$-$
$\dfrac{3x-7}{3-x}$	$-$		$+$		$-$

$\dfrac{3x-7}{3-x} > 0$ if $\dfrac{7}{3} < x < 3$

$\Rightarrow$ Soln. set: $\left(\dfrac{7}{3}, 3\right)$

g_2-) $\dfrac{x-1}{3-x} \leq 2$

$\Leftrightarrow \dfrac{3x-7}{3-x} \leq 0$

$\boxed{\text{Soln. set: } \left(-\infty, \dfrac{7}{3}\right] \cup (3, \infty)}$

Absolute Value:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Properties of Abs. value:

1-) $|a.b| = |a||b|$

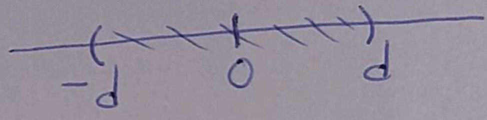
2-) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

3-) $\boxed{|a-b| = |b-a|}$ $\left(= |-(a-b)| = \underbrace{|-1|}_{1}|a-b| = |a-b| \right)$

4-) $-|a| \leq a \leq |a|$, 5-) $|a+b| \leq |a| + |b|$

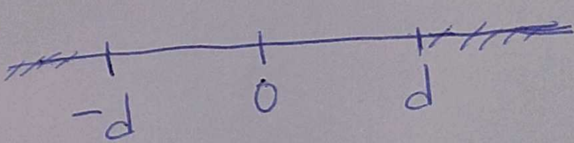
Absolute Value Inequalities: , $d > 0$

1-) $|x| < d \Rightarrow -d < x < d$;



2-) $|x| \leq d \Rightarrow -d \leq x \leq d$

3-) $|x| > d \Rightarrow x < -d$ or $x > d$



4-) $|x| \geq d \Leftrightarrow x \leq -d$ or $x \geq d$

$(-\infty, -d] \cup [d, \infty)$

ex: Solve the following equalities and inequalities:

a.) $|x-4|=1 \Rightarrow$ $x-4=1 \Rightarrow x=5$
or
$-(x-4)=1 \Rightarrow -x+4=1$
$-x=-3 \Rightarrow x=3$

$\left.\right\}$ $x=3$ or $x=5$

Soln. set: $\boxed{\{3, 5\}}$

b.) $\left|\frac{5}{x}\right|=12 \Rightarrow \frac{|5|}{|x|}=12 \Rightarrow \frac{5}{12}=|x| \Rightarrow \begin{cases} x=5/12 \\ \text{or} \\ x=-5/12 \end{cases}$

Soln. set: $\left\{\mp\frac{5}{12}\right\}$

c.) $|4+3x|=6 \Rightarrow$ $4+3x=6 \Rightarrow 3x=6-4=2 \Rightarrow x=\frac{2}{3}$
or
$4+3x=-6 \Rightarrow 3x=-6-4=-10 \Rightarrow x=\frac{-10}{3}$

Soln. set: $\left\{\frac{-10}{3}, \frac{2}{3}\right\}$

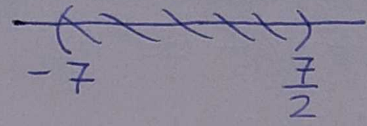
** d.) $|7x+3|=x$ $\Rightarrow$ $\left.\begin{array}{l} 7x+3=x \\ \text{or} \\ 7x+3=-x \end{array}\right\}$ $\begin{array}{l} 6x=-3 \Rightarrow x=-1/2 \\ \text{or} \\ 8x=-3 \Rightarrow x=-3/8 \end{array}$

!! $\boxed{(\Rightarrow x \geq 0)}$
(since $|7x+3| \geq 0$)

$\boxed{\text{but } x \geq 0} \Rightarrow \boxed{\text{No soln.}}$

Soln. set: $\{\emptyset\}$

e-) $|4x+7| < 21 \Rightarrow -21 < 4x+7 < 21 \Rightarrow \frac{-28}{4} < \frac{4x}{4} < \frac{14}{4}$

$\Rightarrow -7 < x < \frac{7}{2} \Rightarrow$  $\Rightarrow \boxed{\left(-7, \frac{7}{2}\right)}$

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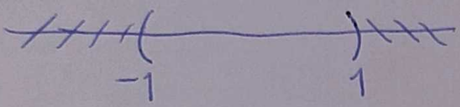
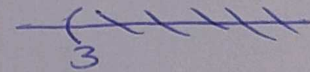
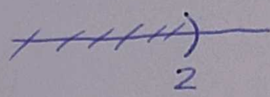
f-) $\frac{1}{|2x-5|} < 1 \Leftrightarrow |\underbrace{2x-5}_{a}| > 1 \Rightarrow$ $2x-5 > 1 \Rightarrow 2x > 6$

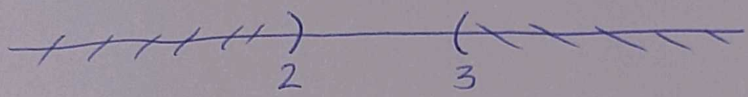
or   or

$2x-5 < -1 \Rightarrow 2x < 4$



$2x > 6 \Rightarrow x > 3 \Rightarrow$ 

$2x < 4 \Rightarrow x < 2 \Rightarrow$ 



Soln. set: $\boxed{(-\infty, 2) \cup (3, \infty)}$

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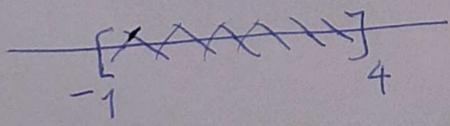
g-) $-2 < |2x-3| \leq 5$

i) $|2x-3| \geq 0 > -2 \Rightarrow$ $\boxed{\text{whatever } x \text{ value is, } |2x-3| > -2}$

ii) $|2x-3| \leq 5 \Rightarrow -5 \leq 2x-3 \leq 5 \Rightarrow -5+3 \leq 2x \leq 5+3$

$\frac{-2}{2} \leq \frac{2x}{2} \leq \frac{8}{2}$



$\boxed{-1 \leq x \leq 4}$

$\Rightarrow$ $\boxed{\text{Soln. set: } [-1, 4]}$

h-) $|x-1|+2x \geq 2 \Rightarrow |x-1| \geq 2-2x$

$\Rightarrow \boxed{x-1 \geq 2-2x} \Rightarrow 3x \geq 3 \Rightarrow x \geq 1$

or

$\boxed{-(x-1) \geq 2-2x} \Longleftrightarrow x-1 \leq -(2-2x)$

$x-1 \leq -2+2x$

$2-1 \leq 2x-x$

$1 \leq x$

$\Big\} \; x \geq 1$

Soln. set: $[1, \infty)$

Check: $x=0$:

$|0-1|+2(0) = |-1|+0 = 1 \not\geq 2$

Exercises:

- $|3x+2| < 8$ (Ans.: $\left(-\frac{10}{3}, 2\right)$)
- $|2x-17| < -4$ (Ans.: No soln. or $\{\emptyset\}$)
- $|1-3x| > 2$ (Ans.: $x < -\frac{1}{3}$ or $x > 1 \Rightarrow \left(-\infty, -\frac{1}{3}\right) \cup (1, \infty)$)
- $|3x-2| + x < \frac{5}{2}$ (Ans.: $-\frac{1}{4} < x < \frac{9}{8} \Rightarrow \left(-\frac{1}{4}, \frac{9}{8}\right)$)
- $\dfrac{(x-3)(x+1)}{(x+2)} > 0$ (Ans.: $(-2,-1) \cup (3, \infty)$)
- $(x+1)^2 \cdot x \cdot (x-1) < 0$ (Ans.: $(0,1)$)
- $\dfrac{x^2-4}{-x^2+5x+6} \geq 0$ (Ans.: $[-2,-1) \cup [2,6)$)

Quadratic Equations:

$\boxed{ax^2+bx+c=0}$ $\xrightarrow{\text{solving}}$ $\boxed{\text{factorization}}$

$(a,b,c\in\mathbb{R},\ a\neq 0)$ → Using quadratic formula:

$$x_{1,2}=\frac{-b\mp\sqrt{b^2-4ac}}{2a}$$

$b^2-4ac=\Delta$ (discriminant)

* $\Delta<0 \Rightarrow$ there is no (real) solution
* $\Delta=0 \Rightarrow$ repeated real roots $x_{1,2}=\frac{-b}{2a}$
* $\Delta>0 \Rightarrow$ there are 2 real, distinct roots.

Examples: Solve the following equations:

1-) $3x^2-5x-2=0 \Rightarrow (3x+1)(x-2)=0 \Rightarrow x=-\frac{1}{3}$ or $x=2$

$\left(x_{1,2}=\frac{5\mp\sqrt{25-4(3)(-2)}}{2(3)}=\frac{5\mp7}{6} \begin{cases} 2 \\ -1/3 \end{cases}\right)$

2-) $3x^2-2x+1=0$ ⟶ there is no (real) factorization

$a=3,\ b=-2,\ c=1$: $\Delta=b^2-4ac=(-2)^2-4(3)(1)=-8<0$

$\Rightarrow$ no real root!

3-) $\frac{1}{x^4}-\frac{9}{x^2}+8=0$ $\boxed{\text{Soln. set: } \left\{\mp1,\ \mp\frac{1}{2\sqrt{2}}\right\}}$

Let $u=\frac{1}{x^2}$: $\boxed{u^2-9u+8=0}$

$(u-8)(u-1)=0$

$u=8 \Rightarrow \frac{1}{x^2}=8 \Rightarrow x^2=\frac{1}{8} \Rightarrow x=\mp\frac{1}{2\sqrt{2}}$

$u=1 \Rightarrow \frac{1}{x^2}=1 \Rightarrow x^2=1 \Rightarrow x=\mp1$

Exercises: Solve the following quadratics.

- $x^2-10x+6=0$ (x_1, x_2: distinct real roots)

- $4x^2+5x-6=0$ (x_1, x_2)

- $3x^2+8x+6=0$ ($\Delta = b^2-4ac = 64-4(3)(6) < 0$ no real roots!)

- $x^2-22x+121=0$ ($x_1 = x_2 = 11$) (or $\Delta = 0$)

  $(x-11)^2=0$

- $x^4-2x^2+3=0$ ($u=x^2$; No real root)

- $(x-3)^2+5(x-3)+4=0$ ($u = x-3 \Rightarrow \{-1, 2\}$)

  $u^2+5u+4=0$

  $(u+4)(u+1)=0$

  $u=-4$ or $u=-1$

  $x-3=-4$ $\quad$ $x-3=-1$

  $x=-1$ $\quad$ $x=2$

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