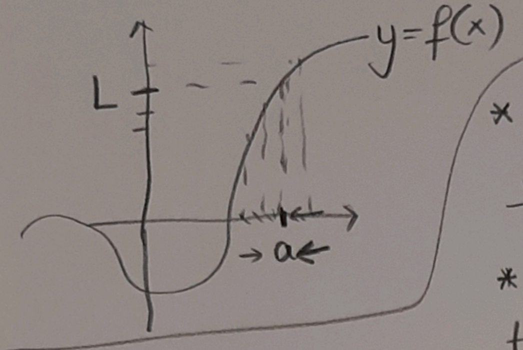
Math 113 ; Nov. 24, 2020

Limits: If the function f approaches L as x approaches "a", we say that the limit of f at "a" is L.
We write;

$$\lim_{x \to a} f(x) = L$$



One-sided limits:

* If x approaches "a" with values larger than a, we say that $x \to a^+$
* If x approaches a with values smaller than a, we say that $x \to a^-$

* If $x \to a^+$, and $f(x) \to L \Rightarrow \lim\limits_{x \to a^+} f(x) = L$ (right-sided limit of f)

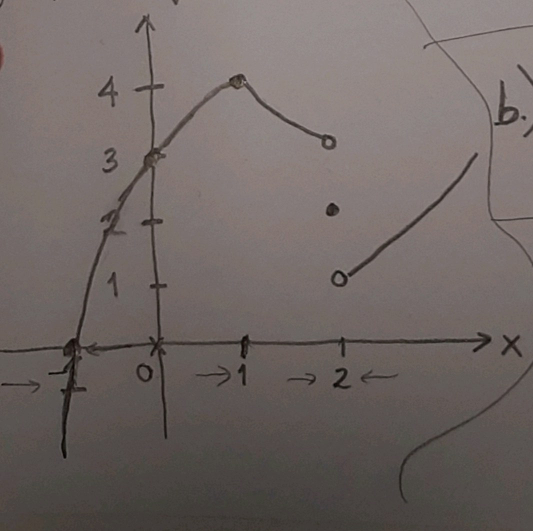
* When $x \to a^-$, and $f(x) \to L \Rightarrow \lim\limits_{x \to a^-} f(x) = L$ (left-sided limit of f)

* $\boxed{\lim\limits_{x \to a} f(x) = L \iff \lim\limits_{x \to a^-} f(x) = L \text{ and } \lim\limits_{x \to a^+} f(x) = L}$

Estimating a limit from a graph:

Examples: Find the following limits and values using the graph of $f(x)$:

1



a.) $f(-1) = 0$, $\lim\limits_{x \to -1^+} f(x) = 0$, $\lim\limits_{x \to -1^-} f(x) = 0 \Rightarrow \lim\limits_{x \to -1} f(x) = 0$

b.) $f(0) = 3$, $\lim\limits_{x \to 0^-} f(x) = 3$, $\lim\limits_{x \to 0^+} f(x) = 3$, $\lim\limits_{x \to 0} f(x) = 3$

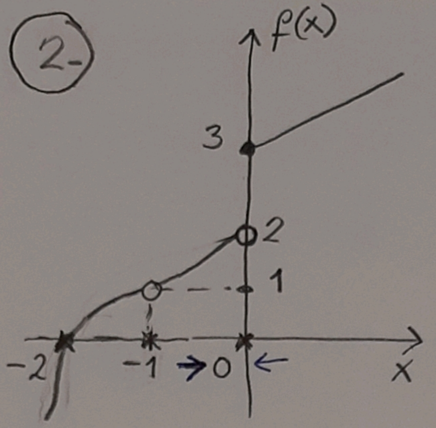
c.) $f(1) = 4$, $\lim\limits_{x \to 1^-} f(x) = 4$, $\lim\limits_{x \to 1^+} f(x) = 4$, $\lim\limits_{x \to 1} f(x) = 4$

d.) $f(2) = 2$, $\lim\limits_{x \to 2^-} f(x) = 3$, $\lim\limits_{x \to 2^+} f(x) = 1 \Rightarrow$

$\Rightarrow \boxed{\lim\limits_{x \to 2} f(x) = \text{D.N.E.}}$

2

2-



a) $f(0)=3$, $\lim\limits_{x\to 0^-} f(x)=2$, $\lim\limits_{x\to 0^+} f(x)=3$

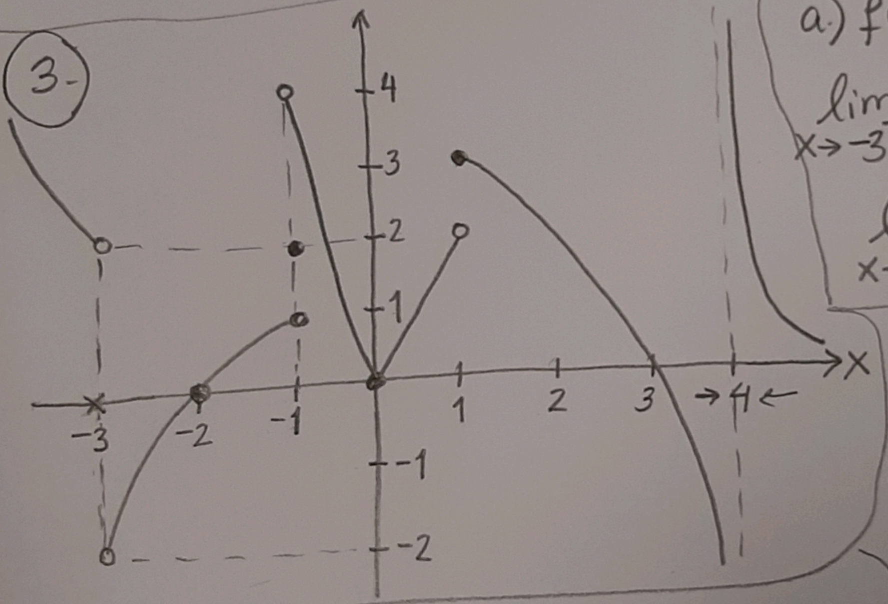
$\Rightarrow \boxed{\lim\limits_{x\to 0} f(x)=\text{D.N.E}}$

b) $f(-1)=\text{undefined}$

$\lim\limits_{\substack{x\to -1^- \\ (x<-1)}} f(x)=1$, $\lim\limits_{\substack{x\to -1^+ \\ (x>-1)}} f(x)=1 \Rightarrow \boxed{\lim\limits_{x\to -1} f(x)=1}$

c) $\boxed{f(-2)=0}$, $\lim\limits_{\substack{x\to -2^- \\ (x<-2)}} f(x)=0$, $\lim\limits_{\substack{x\to -2^+ \\ (x>-2)}} f(x)=0 \Rightarrow \boxed{\lim\limits_{x\to -2} f(x)=0}$

3-



a) $f(-3)=\text{not defined}$

$\lim\limits_{x\to -3^-} f(x)=2$, $\lim\limits_{x\to -3^+} f(x)=-2$

$\lim\limits_{x\to -3} f(x)=\text{D.N.E.}$

b) $f(-2)=0$

$\lim\limits_{x\to -2^-} f(x)=0=\lim\limits_{x\to -2^+} f(x)$

$\Rightarrow \lim\limits_{x\to -2} f(x)=0$

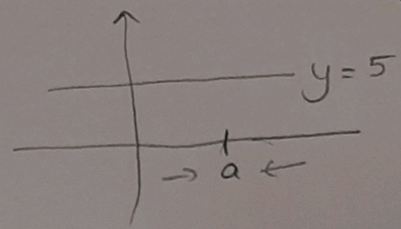
c) $f(-1)=2$, $\lim\limits_{x\to -1^-} f(x)=1$, $\lim\limits_{x\to -1^+} f(x)=4$ $\Rightarrow \lim\limits_{x\to -1} f(x)=\text{D.N.E.}$

d) $f(4)=\text{undefined}$, $\lim\limits_{\substack{x\to 4^- \\ (x<4)}} f(x)=-\infty$, $\lim\limits_{\substack{x\to 4^+ \\ (x>4)}} f(x)=\infty\ (+\infty)$

$\lim\limits_{x\to 4} f(x)=\text{D.N.E.}$

Properties of Limit:

1.) If $f(x)=c \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} c = c$



2.) $\lim_{x\to a} x^n = a^n$, for any positive integer "n" $\left(\lim_{x\to 2} x^3 = 2^3 = 8\right)$

* If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

3.) $\lim_{x\to a} (f(x) \mp g(x)) = L \mp M$

4.) $\lim_{x\to a} (f(x) \cdot g(x)) = L \cdot M$

5.) $\lim_{x\to a} (c \cdot f(x)) = c \cdot L$ (for any constant c)

6.) $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ if $M \neq 0$

7.) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$

8.) If f is a polynomial function $\Rightarrow$

$$\lim_{x\to a} f(x) = f(a).$$

Examples: Evaluate the following limits if they exist

$$\left[\frac{0}{a} = 0 \ (a \neq 0) \qquad \frac{a}{0} = \text{undefined}\right]$$

1.) $\lim\limits_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$

2.) $\lim\limits_{x\to -3} \left(\frac{x-3}{x+5}\right) = \frac{-3-3}{-3+5} = \frac{-6}{2} = -3$

3.) $\lim\limits_{x\to 0} \left(\frac{x}{x^3-4x+3}\right) = \frac{0}{0-0+3} = \frac{0}{3} = 0$

4.) $\lim\limits_{x\to 1} \left(\frac{x^2-1}{x-1}\right) \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} \underset{x\neq 1}{=} 1+1 = 2 \quad (x \neq 1)$

5.) $\lim\limits_{x\to 1} \left(\frac{x^3-1}{x-1}\right) \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} \underset{x\neq 1}{=} 3 \quad (x \neq 1)$

6.) $\lim\limits_{x\to 2} \left(\frac{x^2-5x+6}{x^2-3x+2}\right) \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 2} \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-1)} \underset{\downarrow (x\neq 2)}{=} \frac{2-3}{2-1} = \frac{-1}{1} = -1 \quad (x \neq 2)$

7.) $\lim\limits_{x\to 3} \left(\frac{x-3}{x^2-9}\right) = \frac{1}{6}$ (HW)

8.) $\lim\limits_{x\to 1} \left(\frac{2x^2+3x-5}{x-1}\right) \underset{\downarrow \left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(2x+5)}{\cancel{(x-1)}} \underset{\downarrow x\neq 1}{=} 7 \quad (x \neq 1)$

9.) $\lim\limits_{x\to (-1)} \sqrt{2x^2+3} = \sqrt{2(-1)^2+3} = \sqrt{5}$

10.) $\lim_{x\to 4}\left(\frac{\sqrt{x}-1}{x+1}\right) = \frac{\sqrt{4}-1}{5} = \frac{1}{5}$

11.) $\lim_{x\to e}\left[\ln(4x^2)\right] = \ln(4e^2) = \ln(4) + 2\underbrace{\ln(e)}_{1} = \boxed{\ln 4 + 2}$

12.) $\lim_{x\to 4}\log_2\left[\left(\frac{x}{x-2}\right)^2\right] = \log_2\left[\left(\frac{4}{4-2}\right)^2\right] = \log_2 2^2 = 2\underbrace{\log_2 2}_{1}$

$= 2$

13.) $\lim_{x\to 2}\left(e^x + 2^x + 4x\right) = e^2 + 2^2 + 4(2) = \boxed{e^2 + 12}$

Special limits:

① $\lim_{x\to 0}\left[\frac{x^3+2x+1}{x(2x+5)}\right] = \frac{1}{0(5)} = \boxed{\frac{1}{0}} \Rightarrow$ limit does not exist (d.n.e.)

② $\lim_{x\to 1}\left(\frac{x^2-5x+6}{x^2-3x+2}\right) = \frac{2}{0} = \boxed{\text{d.n.e.}}$

③ $\lim_{x\to 0}\frac{x}{|x|} = ?$: when $x<0 \Rightarrow \lim_{x\to 0^-}\frac{x}{|x|} = \lim_{x\to 0^-}\left(\frac{x}{-x}\right) = \lim_{x\to 0^-}(-1)$

$= \boxed{-1}$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases}$

when $x>0 \Rightarrow \lim_{x\to 0^+}\frac{x}{|x|} = \lim_{x\to 0^+}\left(\frac{x}{x}\right)$

$= \lim_{x\to 0^+}(1) = 1$

$\lim_{x\to 0^-}\frac{x}{|x|} = -1 \neq 1 = \lim_{x\to 0^+}\frac{x}{|x|} \Rightarrow \boxed{\lim_{x\to 0}\frac{x}{|x|} = \text{d.n.e}}$

4 $\lim_{x\to 1} |x-1| = ?$: $x<1 \Rightarrow \lim_{x\to 1^-} |x-1| = \lim_{x\to 1^-} (-(x-1)) = 0$

$x>1 \Rightarrow \lim_{x\to 1^+} |x-1| = \lim_{x\to 1^+} (x-1) = 0$

$$\Rightarrow \boxed{\lim_{x\to 1} |x-1| = 0}$$

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5 $f(x) = \begin{cases} x^2+1 & \text{if } x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$,

a) $\lim_{x\to -1} f(x) = ?$

b) $\lim_{x\to 2} f(x) = ?$

c) $\lim_{x\to 1} f(x) = ?$

a.) $\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2+1) = (-1)^2+1 = 2$

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b.) $\lim_{x\to 2} f(x) = \lim_{x\to 2} (x-1) = 2-1 = 1$

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c.) $\lim_{\substack{x\to 1^- \\ (x<1)}} f(x) = \lim_{x\to 1^-} (x^2+1) = 1^2+1 = 2$ (left limit at $x=1$)

$\lim_{\substack{x\to 1^+ \\ (x>1)}} f(x) = \lim_{x\to 1^+} (x-1) = 1-1 = 0$ (right limit at $x=1$)

$\lim_{x\to 1^-} f(x) = 2 \neq 0 = \lim_{x\to 1^+} f(x) \Rightarrow \boxed{\lim_{x\to 1} f(x) = \text{d.n.e.}}$

6

$$f(x)=\begin{cases} 2-x^2, & x>1 \\ -2+3x, & 0\le x\le 1 \\ 4-x^2, & x<0 \end{cases}$$

a) $\lim\limits_{x\to 1} f(x)=?$: $\lim\limits_{\substack{x\to 1^- \\ (x<1)}} f(x)=\lim\limits_{x\to 1^-}(-2+3x)=-2+3(1)=1$

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$\lim\limits_{\substack{x\to 1^+ \\ (x>1)}} f(x)=\lim\limits_{x\to 1^+}(2-x^2)=2-1=1$

$\Rightarrow \boxed{\lim\limits_{x\to 1} f(x)=1}$ (since right and left limits of $f(x)$ at $x=1$ are equal and $=1$).

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b) $\lim\limits_{x\to 0} f(x)=?$: $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x)=\lim\limits_{x\to 0^-}(4-x^2)=4$

$\not\parallel$

$\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x)=\lim\limits_{x\to 0^+}(-2+3x)=-2$

$\Rightarrow \lim\limits_{x\to 0} f(x)=$ d.n.e. (since right and left limits at $x=0$ are not equal).

7.) $$\lim_{x\to 0}\left(\frac{\sqrt{x+9}-3}{x}\right) \underset{\left(\frac{0}{0}\right)}{\overset{}{=}} \lim_{x\to 0}\left[\frac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x(\sqrt{x+9}+3)}\right]$$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{x\to 0}\left[\frac{\cancel{x}+\cancel{9}-\cancel{9}}{\cancel{x}(\sqrt{x+9}+3)}\right] \underset{x\neq 0}{=} \lim_{x\to 0}\left[\frac{1}{\sqrt{x+9}+3}\right] = \frac{1}{6} \quad (x\neq 0)$$

$\left(\frac{x}{x}=1\right)$

8.) $$\lim_{x\to 0}\left(\frac{x}{\sqrt{25+x}-5}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}\left[\frac{x(\sqrt{25+x}+5)}{(\sqrt{25+x}-5)(\sqrt{25+x}+5)}\right] \underset{x\neq 0}{=} 10$$

(HW)

9.) $$\lim_{x\to 1}\left(\frac{\sqrt{15+x}-4}{x-1}\right) \underset{\left(\frac{0}{0}\right)}{=} \cdots\cdots \underset{(x\neq 1)}{=} \lim_{x\to 1}\left(\frac{1}{\sqrt{15+x}+4}\right) = \frac{1}{8}$$

HW

10) $$\lim_{x\to 5}\left(\frac{\sqrt{x-1}-2}{x^2-6x+5}\right) \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 5}\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\cancel{(x-5)}(x-1)(\sqrt{x-1}+2)} \underset{x\neq 5}{=} \frac{1}{16}$$

$\cancel{(x-5)}$

HW

11.) $$\lim_{x\to 2}\left(\frac{\sqrt{3x+10}-4}{x-2}\right) \underset{\left(\frac{0}{0}\right)}{=} \cdots\cdots = \frac{3}{8} \quad (x\neq 2)$$

HW