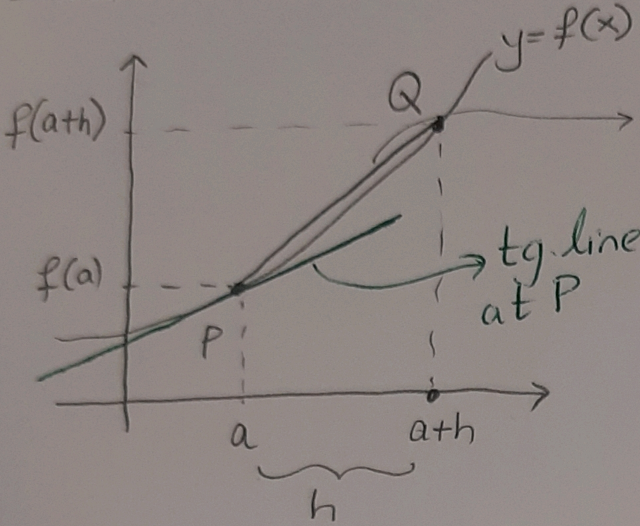
Math 113; Dec. 8, 2020

Differentiation:



$$m_{\text{secant line}} = \frac{f(a+h)-f(a)}{(a+h)-a}$$

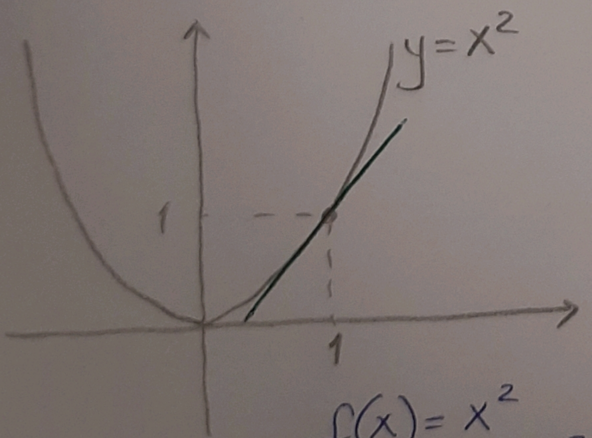
$$m_{sec} = \frac{f(a+h)-f(a)}{h}$$

As $h \to 0 \Rightarrow$ secant line joining pts. P & Q becomes the tangent line drawn to $y=f(x)$ at the pt. P

$$\Rightarrow \boxed{m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$: slope of the tang. line at P.

Examples: Find the slope of the tangent line of the following functions at the given pts.:

① $f(x)=x^2$ at $x=1$:



$f(x)=x^2$

$f(1+h)=(1+h)^2$

$$m = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^2-(1)^2}{h} = \lim_{h\to 0}\frac{\cancel{1}+2h+h^2-\cancel{1}}{h}$$

$$= \lim_{\substack{h\to 0\\(h\neq 0)}}\frac{\cancel{h}(2+h)}{\cancel{h}} \underset{h\neq 0}{=} \lim_{h\to 0}(2+h) = 2$$

slope = 2

② $f(x)=x^3+3$ at $x=-2$:

$$m = \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}\frac{[(-2+h)^3+3]-[(-2)^3+3]}{h}$$

$$= \lim_{h\to 0}\frac{(-2)^3+3(-2)^2(h)+3(-2)(h)^2+(h)^3+\cancel{3}-(-8)\cancel{-3}}{h}$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$= \lim_{h\to 0}\frac{\cancel{-8}+12h-6h^2+h^3\cancel{+8}}{h} = \lim_{\substack{h\to 0\\(h\neq 0)}}\frac{\cancel{h}(12-6h+h^2)}{\cancel{h}} = \boxed{12}$$

Derivative of a function:

* The derivative of a function f is the function denoted by f' and defined by:

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$ ← Definition of the derivative of $f(x)$

if this limit exists.

* We will use f', $f'(x)$, $\frac{df}{dx}$, to denote the derivative.

Examples:

① $f(x)=x^2$:

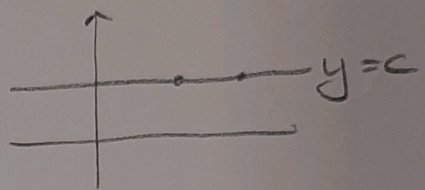
$$f'(x) = \boxed{(x^2)'} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h} = \lim_{\substack{h\to 0\\(h\neq 0)}}\frac{\cancel{h}(2x+h)}{\cancel{h}} = \boxed{2x}$$

2. $g(x) = x^3 \Rightarrow \frac{d}{dx}(x^3) = (x^3)' = ?$

$$g'(x) = (x^3)' = \lim_{h \to 0} \left[\frac{(x+h)^3 - x^3}{h} \right] = \lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

$$= \lim_{\substack{h \to 0 \\ (h \neq 0)}} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = \boxed{3x^2}$$



Rules of Differentiation:

1) $\frac{d(c)}{dx} = 0$, for any constant "c" $\quad ((c)' = 0)$

2) $\boxed{\frac{d(x^n)}{dx} = n x^{n-1}}$ $(n \in \mathbb{R})$ ← Power rule

3) $\frac{d(c f(x))}{dx} = c \frac{d(f(x))}{dx}$ for any constant "c". $\quad ((c f)' = c f')$

4) $(f(x) \mp g(x))' = f'(x) \mp g'(x)$

5) $(f(x) \cdot g(x))' = \boxed{f'(x) \cdot g(x) + f(x) \cdot g'(x)}$ ← Product rule

6) $\left(\frac{f(x)}{g(x)} \right)' = \boxed{\frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}}$ ← Quotient rule

Examples: Evaluate the derivatives of the following functions:

① $f(x) = x^4 - \sqrt[3]{x} + 3 = x^4 - x^{1/3} + 3 \Rightarrow f'(x) = 4x^3 - \frac{1}{3}x^{\frac{1}{3}-1} + 0$

$\Rightarrow \boxed{f'(x) = 4x^3 - \frac{1}{3\sqrt[3]{x^2}}}$

② $y(x) = \frac{x^3}{3} - \frac{3}{x^3} = \frac{1}{3}x^3 - 3 \cdot x^{-3} \Rightarrow y'(x) = \frac{1}{\cancel{3}}(\cancel{3}x^2) - 3 \cdot (-3x^{-4})$

$\Rightarrow y'(x) = x^2 + 9x^{-4} = \boxed{x^2 + \frac{9}{x^4}}$

③ $g(x) = \frac{3}{\sqrt[4]{x^3}} + \frac{2x^2}{\sqrt{x}} = 3 \cdot x^{-3/4} + 2x^{2-\frac{1}{2}} = 3x^{-3/4} + 2x^{3/2}$

$\Rightarrow g'(x) = 3\left(-\frac{3}{4}x^{-\frac{3}{4}-1}\right) + \cancel{2}\left(\frac{3}{\cancel{2}}x^{\frac{3}{2}-1}\right) = -\frac{9}{4}x^{-\frac{7}{4}} + 3x^{1/2}$

$= \boxed{-\frac{9}{4\sqrt[4]{x^7}} + 3\sqrt{x}}$

④ $f(x) = \frac{7x^3 + x}{6\sqrt{x}} = \frac{7}{6}\left(\frac{x^3}{\sqrt{x}}\right) + \frac{1}{6}\left(\frac{x}{\sqrt{x}}\right) = \frac{7}{6}x^{5/2} + \frac{1}{6}x^{1/2}$

$\Rightarrow f'(x) = \frac{7}{6}\left(\frac{5}{2}x^{\frac{5}{2}-1}\right) + \frac{1}{6}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$

$= \frac{35}{12}x^{3/2} + \frac{1}{12}x^{-1/2} = \boxed{\frac{35}{12}\sqrt{x^3} + \frac{1}{12\sqrt{x}}}$

5. $y = x^2(x^3-2x) = x^5 - 2x^3$

$y' = 5x^4 - 6x^2$ (obtain the same answer using product rule)

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6. $f(x) = (7x^3+14x^2-6)(x^8-18)$

Using product rule: $f'(x) = (7x^3+14x^2-6)' \cdot (x^8-18) + (7x^3+14x^2-6)\cdot(x^8-18)'$

$$\Rightarrow f'(x) = (21x^2+28x)\cdot(x^8-18) + (7x^3+14x^2-6)(8x^7)$$

$$= \boxed{21x^{10} - 378x^2 + 28x^9 - (28)(18)x + 56x^{10} + 112x^9 - 48x^7}$$

Simplify!

HW: $f(x) = (x^2+4x^3)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) \Rightarrow f'(x) = ?$

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7. $y = 1 - \frac{5}{2x+5} + \frac{2x}{3x+1} \Rightarrow y' = (1)' - \left(\frac{5}{2x+5}\right)' + \left(\frac{2x}{3x+1}\right)'$

$$\Rightarrow y' = 0 - \frac{\cancel{(0)(2x+5)} - (2)(5)}{(2x+5)^2} + \frac{(2)(3x+1) - (3)(2x)}{(3x+1)^2}$$

$$= -\left[\frac{-10}{(2x+5)^2}\right] + \frac{\cancel{6x} + 2 - \cancel{6x}}{(3x+1)^2} = \boxed{\frac{10}{(2x+5)^2} + \frac{2}{(3x+1)^2}}.$$

$$\boxed{\left[\frac{c}{f(x)}\right]' = \frac{-c \cdot f'(x)}{(f(x))^2}}$$

eg.

$$\left(\frac{-3}{x^2+5x}\right)' = \frac{-(-3)(2x+5)}{(x^2+5x)^2} = \boxed{\frac{3(2x+5)}{(x^2+5x)^2}}$$

8 $f(x) = \dfrac{(9x-1)(3x+2)}{(4-5x)} = \dfrac{27x^2+18x-3x-2}{(4-5x)} = \dfrac{27x^2+15x-2}{(4-5x)}$

$$f'(x) = \frac{(54x+15)(4-5x)-(-5)(27x^2+15x-2)}{(4-5x)^2}$$

quotient rule

$$= \frac{216x-270x+60-\cancel{75x}+135x^2+\cancel{75x}-10}{(4-5x)^2}$$

$$= \boxed{\frac{135x^2-54x+50}{(4-5x)^2}}$$

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9 $y=(2x+3)(x^7-4x^2)(1+x+x^2)$

$y' = (2x+3)' \cdot \left[(x^7-4x^2)(1+x+x^2)\right] + (2x+3) \cdot \left[(x^7-4x^2)(1+x+x^2)\right]'$

$= (2)(x^7+x^8+x^9-4x^2-4x^3-4x^4) + (2x+3)\left[(7x^6-8x)(1+x+x^2)+(x^7-4x^2)(1+2x)\right]$

$= 2x^7+2x^8+2x^9-8x^2-8x^3-8x^4+(2x+3)\left[7x^6+7x^7+7x^8-8x-8x^2-8x^3 + x^7-4x^2+2x^8-8x^3\right]$

$= 2x^7+2x^8+2x^9-8x^2-8x^3-8x^4+(2x+3)\left[9x^8+8x^7+7x^6-16x^3-12x^2-8x\right]$

= (finish this!)

Derivatives of Logarithms and Exponentials:

① $\boxed{f(x) = \ln x} \Rightarrow \boxed{f' = \frac{1}{x}}$

Proof: $(\ln x)' = \lim\limits_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim\limits_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \left(\frac{0}{0}\right)$

L'Hospital's rule: $\Rightarrow \boxed{\frac{1}{x}}$

② $\boxed{f(x) = \log_b x} \Rightarrow f'(x) = \left(\frac{\ln x}{\ln b}\right)' = \left(\frac{1}{\ln b}\right) \cdot \frac{1}{x}$

$\begin{pmatrix} b \neq 1 \\ b > 0 \end{pmatrix}$ $\Rightarrow \boxed{f'(x) = \frac{1}{x \cdot \ln b}}$

③ $f(x) = e^x \Rightarrow \boxed{f'(x) = e^x}$

④ $f(x) = b^x \Rightarrow \boxed{f'(x) = b^x \cdot \ln b}$

$(b \neq 1, b > 0)$

Examples: Find the derivatives of the following functions:

① $y = 3e^x + x^3 + \sqrt[5]{x^2} + 4\ln x + 5$

$y' = 3e^x + 3x^2 + x^{\frac{2}{5}-1} + 4 \cdot \frac{1}{x} + 0 = \boxed{3e^x + 3x^2 + \frac{1}{\sqrt[5]{x^3}} + \frac{4}{x}}$

2) $y=(x^2+x+1)(x^3+e^x+\ln x)$

$y'=(2x+1)(x^3+e^x+\ln x)+(x^2+x+1)\left(3x^2+e^x+\frac{1}{x}\right)$

$=2x^4+2xe^x+2x\ln x+x^3+e^x+\ln x$
$+3x^4+x^2e^x+x+3x^3+xe^x+1+3x^2+e^x+\frac{1}{x}$

$=$ (simplify)

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3) $f(x)=x^4\cdot 2^x \Rightarrow f'(x)=(4x^3)(2^x)+(x^4)\cdot(2^x\cdot\ln 2)$

$=\boxed{2^x\cdot x^3(4+x\cdot\ln 2)}$

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4) $f(x)=\dfrac{3^x}{1+x^2} \Rightarrow f'(x)=\dfrac{(3^x\cdot\ln 3)(1+x^2)-(2x)(3^x)}{(1+x^2)^2}$

$\Rightarrow f'(x)=\dfrac{3^x\cdot[\ln 3+\ln 3\cdot x^2-2x]}{(1+x^2)^2}$

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5) $y=(x^3+5)\cdot\log_5 x \Rightarrow y'=(3x^2)\cdot\log_5 x+(x^3+5)\cdot\dfrac{1}{x\cdot\ln 5}$

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6) $y=\dfrac{\log_7 x+x^3}{x^2+x+8} \Rightarrow y'=\dfrac{\left(\frac{1}{x\cdot\ln 7}+3x^2\right)(x^2+x+8)-(\log_7 x+x^3)(2x+1)}{(x^2+x+8)^2}$

<u>Question</u>?: $y=\log_7(x^2+5x-\sqrt{x}) \Rightarrow y'=?$

$\boxed{\text{Chain rule}}$!!!

(NEXT WEEK)