Math 113 ; Dec. 29, 2020

Matrix Operations, Matrix Multiplication:

$$A=\begin{bmatrix} 1 & 6 & 0 & 4 \\ 2 & 3 & 1 & 0 \\ 4 & 7 & 2 & 8 \end{bmatrix}$$

→ rows of matrix A: an array of numbers

↓ columns of matrix A

→ matrix A has $\boxed{3 \text{ rows}}$ and $\boxed{4 \text{ columns}}$ ⇒

matrix A is of size (or order) : $\boxed{3\times 4}$

* Each number in the matrix A is called the $\boxed{\text{entry}}$ at the i^{th} row/ j^{th} column position : $\boxed{a_{ij}}$

eg. $a_{23}=1$, $a_{34}=8$, $a_{12}=6$, . . .

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a) determine $a_{32}=7$

b) the order (size) of A: 3×4

Let $k\in\mathbb{R}$. Then the multiplication of a matrix $A=[a_{ij}]_{n\times m}$

n: rows

m: columns

by a constant $k\in\mathbb{R}$:

is determined by multiplying each entry of A by the same constant $k\in\mathbb{R}$:

$$3A=\begin{bmatrix} 3\times1 & 3\times6 & 3\times0 & 3\times4 \\ 3\times2 & 3\times3 & 3\times1 & 3\times0 \\ 3\times4 & 3\times7 & 3\times2 & 3\times8 \end{bmatrix}=\begin{bmatrix} 3 & 18 & 0 & 12 \\ 6 & 9 & 3 & 0 \\ 12 & 21 & 6 & 24 \end{bmatrix}=[3a_{ij}]_{3\times4}$$

$i=1,2,3$

$j=1,2,3,4$

Sum of two matrices: Matrices of the same order (same size) can be added by adding the respective entries of the 2 matrices at the same position:

eg.: $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} -1 & 4 & 7 \\ 0 & 5 & -3 \end{bmatrix}_{2\times 3}$

$$A+B = \begin{bmatrix} 2+(-1) & 3+4 & 4+7 \\ -1+0 & 0+5 & 5+(-3) \end{bmatrix} = \begin{bmatrix} 1 & 7 & 11 \\ -1 & 5 & 2 \end{bmatrix}_{2\times 3}$$

$C = \begin{bmatrix} -1 & 0 \\ 4 & 5 \\ 7 & -3 \end{bmatrix}_{3\times 2}$

A is of order: 2×3
C " " " : 3×2

$A+C$ or $C+A$ are not defined

Difference of matrices: $A - B = A + (-1)B$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 7 \\ 0 & 5 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -4 & -7 \\ 0 & -5 & 3 \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} 3 & -1 & -3 \\ -1 & -5 & 8 \end{bmatrix}}_{A-B}$$

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eg. Write $A = [a_{ij}]$ if A is 2×2 and $\boxed{a_{ij} = 2i + j}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2(1)+1 & 2(1)+2 \\ 2(2)+1 & 2(2)+2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

eg. Write $A=[a_{ij}]$ if A is 2×3 and $a_{ij}=2i+j$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=\begin{bmatrix} 2(1)+1 & 2(1)+2 & 2(1)+3 \\ 2(2)+1 & 2(2)+2 & 2(2)+3 \end{bmatrix}=\begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$$

Equal matrices: Two matrices of equal (same) orders are said to be equal iff the corresponding entries are equal at the same positions of the two matrices.

eg. Determine the values of x & y if:

$$\begin{bmatrix} x & y+2 \\ x+y & 4 \end{bmatrix}=\begin{bmatrix} x & 1 \\ 0 & 4 \end{bmatrix} \iff$$

$x = x$ ✓

$y+2=1 \Rightarrow \boxed{y=-1}$

$\boxed{x+y=0}$

$4=4$

$\boxed{y=-1} \Rightarrow \boxed{x=1}$

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eg. If $\begin{bmatrix} 3x & -y \\ x & y \end{bmatrix}=\begin{bmatrix} 3 & 2 \\ 2x & y \end{bmatrix}$, then $y=?$

$3x=3$

$-y=2 \rightarrow \boxed{y=-2}$

$x=2x$

$y=y$

$\boxed{x=1}$

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eg. If $4\begin{bmatrix} 1 & x \\ -2 & 0 \end{bmatrix}+2\begin{bmatrix} -2 & 0 \\ y & 0 \end{bmatrix}=\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{}$ then $x=?$, $y=?$

Zero matrix

matrix equation

$$\begin{bmatrix} 4-4 & 4x+0 \\ -8+2y & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 4x \\ -8+2y & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff$$

$4x=0$

$\Rightarrow \boxed{x=0}$

$-8+2y=0$

$\boxed{y=4}$

A square matrix: is one in which the # of rows and # of columns are equal.

i) Identity matrix: $\begin{cases} a_{ij}=1 & \text{if } i=j \\ a_{ij}=0 & \text{if } i\neq j \end{cases}$ and $i,j=1,\dots,n$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{2x2 Identity matrix}} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

(main) diagonal entries

ii) Diagonal matrix: is a square matrix for which every entry except the main diagonal are zero.

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \dots$$

iii) Triangular matrix: there may be some non-zero entries either above or below the main diagonal;

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \text{upper triangular matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \text{lower triangular matrix}$$

eg. Solve the matrix equation:

$$x\begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1} + 2\begin{bmatrix}3\\5\\1\end{bmatrix} + y\begin{bmatrix}0\\3\\0\end{bmatrix} = \begin{bmatrix}4\\3\\x-y-3\end{bmatrix}$$

column matrix

$$\begin{bmatrix}x+6+0\\2x+10+3y\\3x+2+0\end{bmatrix} = \begin{bmatrix}4\\3\\x-y-3\end{bmatrix} \Rightarrow \begin{bmatrix}x+6\\2x+3y+10\\3x+2\end{bmatrix}_{3\times1} = \begin{bmatrix}4\\3\\x-y-3\end{bmatrix}_{3\times1}$$

$$\Leftrightarrow \quad x+6=4 \Rightarrow \boxed{x=-2}$$

$$2x+3y+10=3 \Rightarrow \boxed{3y} = 3-10-2x = -7-2(-2) = \boxed{-3} \Rightarrow \boxed{y=-1}$$

$$\left.\begin{matrix}x=-2\\y=-1\end{matrix}\right\} \overset{?}{\Rightarrow} 3x+2 \overset{?}{=} x-y-3$$

$$\underbrace{3(-2)+2}_{-4} \overset{?}{=} \underbrace{-2-(-1)-3}_{-4} \checkmark$$

Soln.: $\boxed{\begin{matrix}x=-2\\y=-1\end{matrix}}$

Transpose of a Matrix: the matrix B obtained from $A=[a_{ij}]$ by interchanging the rows & columns of A is called the transpose matrix of A and denoted by:

$$\left.\begin{matrix}A^T = B = [a_{ji}]\\ A=[a_{ij}]\end{matrix}\right\}$$

eg.: $A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}2 & 3\\-1 & 4\end{bmatrix}$

eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 2 & -1 & -2 \\ 1 & -3 & -2 \end{bmatrix}_{2\times 3}$, $(A+B)^T = ?$

$$A+B = \begin{bmatrix} 1+2 & 2+(-1) & 3+(-2) \\ 3+1 & 4+(-3) & 5+(-2) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 4 & 1 & 3 \end{bmatrix}_{2\times 3} \checkmark$$

$$(A+B)^T = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2} \checkmark$$

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eg. $$\Bigg(\underbrace{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{3\times 3} + \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}_{2\times 3}}_{\text{not possible}}\Bigg)^T \longrightarrow \text{undefined} \times$$

Trace of a (diagonal) matrix A: trace(A)

is the sum of the diagonal entries.

eg. $\text{trace}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} = 1+(-1)+(-4) = \boxed{-4}$

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eg. $\text{trace}\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix} = 3+(-2)+(2) = \boxed{3}$ $\checkmark$

$$\text{trace}\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}^T = \text{trace}\begin{bmatrix} 3 & 0 & 3 \\ -2 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 3+(-2)+(2) = \boxed{3}$$

Product of two matrices:

$$A_{m\times n} \cdot B_{n\times p} = [A.B]_{\boxed{m\times p}}$$

# of rows: m
# of columns: n

# of rows: n
# of columns: p

But if $p \neq m$:

$$B_{n\times p} \cdot A_{m\times n} : \text{ is undefined}$$

Product of 2 matrices is possible iff # of columns of the first matrix is the same as the # of rows of the second matrix.

ex:

$$\underset{2\times 3}{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 3 & -2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}} = \begin{bmatrix} (3)(3)+(-2)(0)+(1)(3) & (3)(-2)+(-2)(-2)+(1)(1) \\ (0)(3)+(-2)(0)+(1)(3) & (0)(-2)+(-2)(-2)+(1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -1 \\ 3 & 5 \end{bmatrix}_{2\times 2}$$

$$\underset{3\times 2}{\begin{bmatrix} 3 & -2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}} \cdot \underset{2\times 3}{\begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}} = \begin{bmatrix} (3)(3)+(-2)(0) & (3)(-2)+(-2)(-2) & (3)(1)+(-2)(1) \\ (0)(3)+(-2)(0) & (0)(-2)+(-2)(-2) & (0)(1)+(-2)(1) \\ (3)(3)+(1)(0) & (3)(-2)+(1)(-2) & (3)(1)+(1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -2 & 1 \\ 0 & 4 & -2 \\ 9 & -8 & 4 \end{bmatrix}_{3\times 3}$$