Math 113 ; Dec. 21, 2020

Review for Midterm II:

* Optimization Problems ARE NOT INCLUDED!
* Logarithmic Differentiation (12.5) IS INCLUDED!

2019/M2:

1 $$f(x)=\begin{cases} e^x+3-x^2, & x<0 \\ 3x-2, & 0\le x<1 \\ 2-x^2+\ln x, & x\geq 1 \end{cases}$$

Except for $x=0, x=1$, the given function is continuous at the related x-values.

at $x=0$: $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x)=\lim\limits_{x\to 0^-}(e^x+3-x^2)=e^0+3-0=4$

$\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x)=\lim\limits_{x\to 0^+}(3x-2)=3(0)-2=-2$

$\lim\limits_{x\to 0} f(x)=$ d.n.e $\Downarrow$

$\boxed{f \text{ is discont. at } x=0}$

at $x=1$: $f(1)=2-1^2+\ln 1=2-1+0=1$

$\lim\limits_{\substack{x\to 1^- \\ (x<1)}} f(x)=\lim\limits_{x\to 1^-}(3x-2)=3(1)-2=1$

$\lim\limits_{\substack{x\to 1^+ \\ (x>1)}} f(x)=\lim\limits_{x\to 1^+}(2-x^2+\ln x)=2-1+\underbrace{\ln 1}_{0}=1$

$\Rightarrow \lim\limits_{x\to 1} f(x)=1=f(1)$

$\Rightarrow f$ is cont. at $x=1$

The only discont. pt. of $f(x)$ is $\boxed{x=0}$

2.) Eqn. of tg. line: $y-y_0 = m(x-x_0)$

$m \to y'(x_0)$

implicit function

$\boxed{y^3 = x^2 - \ln(xy+1) - 8}$ at the pt. where $x_0 = 0$

$\underline{x=0}$: $y^3 = 0 - \ln(0+1) - 8 = -8 \Rightarrow y^3 = -8 \Rightarrow \boxed{y=-2}$

$\boxed{(x_0, y_0) = (0,-2)}$

$$3y^2 \cdot y' = 2x - \frac{1}{xy+1} \cdot (1 \cdot y + x \cdot y') = 2x - \frac{y}{xy+1} - \frac{x}{xy+1} \cdot y'$$

$$y'\left(3y^2 + \frac{x}{xy+1}\right) = 2x - \frac{y}{xy+1} \Rightarrow y' = \frac{2x - \frac{y}{xy+1}}{3y^2 + \frac{x}{xy+1}}$$

$$\Rightarrow y' = \left.\frac{\frac{2x(xy+1) - y}{\cancel{xy+1}}}{\frac{3y^2(xy+1) + x}{\cancel{xy+1}}}\right|_{(0,-2)} = \frac{\cancel{2(0)(0(-2)+1)} - (-2)}{3(-2)^2(0+1) + 0} = \frac{2}{12} = \frac{1}{6}$$

slope of tg. line

$$y - (-2) = \left(\frac{1}{6}\right)(x - 0) = \frac{1}{6}x$$

$\boxed{y = \frac{1}{6}x - 2}$ eqn. of tg. line

3) Find the derivatives;

a.) $f(x) = \log_{x^2}\left(\sqrt{x^2-5x}\right) \underset{\text{change of base}}{=} \dfrac{\ln(x^2-5x)^{1/2}}{\ln x^2} = \dfrac{\frac{1}{2}\ln(x^2-5x)}{2\ln x}$

$$f'(x) = \frac{1}{4}\left[\frac{\left(\frac{2x-5}{x^2-5x}\right)(\ln x) - \left(\frac{1}{x}\right)(x^2-5x)}{(\ln x)^2}\right]$$

$$= \boxed{\frac{1}{4}\left[\frac{(\ln x)(2x-5) - (x-5)(x^2-5x)}{(x^2-5x)(\ln x)^2}\right]}$$

b.) $y = \left(\dfrac{1+x}{(1+x^3)e^x}\right)^{x^2} \Rightarrow \ln y = \ln\left(\dfrac{1+x}{(1+x^3)e^x}\right)^{x^2} = x^2 \cdot \ln\left(\dfrac{(1+x)}{(1+x^3)e^x}\right)$

$$\Rightarrow \ln y = x^2 \cdot \left[\ln(1+x) - \ln(1+x^3) - \underbrace{\ln e^x}_{x}\right]$$

$$\ln y = x^2 \cdot \left[\ln(1+x) - \ln(1+x^3) - x\right].$$

$$\frac{y'}{y} = (2x)\left[\ln(1+x) - \ln(1+x^3) - 1\right] + (x^2)\left[\frac{1}{1+x} - \frac{3x^2}{1+x^3} - 1\right]$$

$$\Rightarrow \boxed{y' = \left(\frac{1+x}{(1+x^3)e^x}\right)^{x^2} \cdot \left\{(2x)\left[\ln(1+x) - \ln(1+x^3) - 1\right] + x^2\left[\frac{1}{1+x} - \frac{3x^2}{1+x^3} - 1\right]\right\}}$$

c) $g(x) = \dfrac{(x+2)^3(3x^2+7)\sqrt{x^2+4}}{(x^2+5x-11)(2x+1)^5} \Rightarrow g'(x) = ?$

$$\ln g(x) = \left[3\ln(x+2) + \ln(3x^2+7) + \frac{1}{2}\ln(x^2+4)\right] - \left[\ln(x^2+5x-11) + 5\ln(2x+1)\right]$$

$$\frac{g'(x)}{g(x)} = 3\frac{1}{x+2} + \frac{6x}{3x^2+7} + \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+4} - \frac{2x+5}{x^2+5x-11} - 5\frac{2}{2x+1}$$

$$g'(x) = [g(x)] \cdot \left[\frac{3}{x+2} + \frac{6x}{3x^2+7} + \frac{x}{x^2+4} - \frac{2x+5}{x^2+5x-11} - \frac{10}{2x+1}\right]$$

4.) a) $xe^y = \ln(x-y) + 5$ $\qquad y' = ?$

Use implicit diff.:

$$1.e^y + x.e^y.y' = \frac{1-y'}{x-y} + 0 = \frac{1}{x-y} - \frac{1}{x-y}.y'$$

$$y'\left[xe^y + \frac{1}{x-y}\right] = \frac{1}{x-y} - e^y$$

$$\boxed{y'} = \frac{\dfrac{1-e^y(x-y)}{\cancel{(x-y)}}}{\dfrac{x.e^y(x-y)+1}{\cancel{(x-y)}}} = \boxed{\frac{1-e^y(x-y)}{1+xe^y(x-y)}}$$

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b) $y^x = x^{y^2}$, $y' = ?$

$$\ln(y^x) = \ln(x^{y^2}) \Rightarrow (x)(\ln y) = (y^2)(\ln x)$$

$$1 \cdot \ln y + x \cdot \frac{y'}{y} = (2y \cdot y')(\ln x) + y^2 \cdot \frac{1}{x}$$

$$y'\left[\frac{x}{y} - 2y \cdot \ln x\right] = \frac{y^2}{x} - \ln y$$

$$y' = \frac{\dfrac{y^2 - x\ln y}{x}}{\dfrac{x - 2y^2 \ln x}{y}} = \boxed{\frac{y^3 - xy\ln y}{x^2 - 2xy^2 \ln x}}$$

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c) $y \ln x + x \ln y = x^2 y^2$

$$\left(y' \cdot \ln x + y \cdot \frac{1}{x}\right) + \left(1 \cdot \ln y + x \cdot \frac{y'}{y}\right) = 2x \cdot y^2 + x^2 \cdot 2y \cdot y'$$

$$y'\left[\ln x + \frac{x}{y} - 2x^2 y\right] = 2xy^2 - \frac{y}{x} - \ln y$$

$$\boxed{y' = \frac{2xy^2 - \frac{y}{x} - \ln y}{-2x^2 y + \frac{x}{y} + \ln x}}$$

5) $f(1)=0,\ f'(1)=3,\ f(2)=-1,\ f'(2)=-2,\ g(1)=4,\ g'(1)=2,$
$g(2)=1$ and $g'(2)=5$

a) If $h(x)=\dfrac{2+f(x)}{g(x)-x^2+1} \Rightarrow h'(1)=?$

$$h'(x)=\frac{(f'(x))(g(x)-x^2+1)-(g'(x)-2x)(2+f(x))}{(g(x)-x^2+1)^2}$$

$$\boxed{h'(1)}=\frac{(f'(1))(g(1)-\cancel{1^2}+\cancel{1})-(g'(1)-2(1))(2+f(1))}{(g(1)-\cancel{1^2}+\cancel{1})^2}$$

$$=\frac{(3)(4)-\cancel{(2-2)}^{0}(2+0)}{(4)^2}=\frac{12}{16}=\frac{3}{4}$$

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b) $k(x)=f(g(2x)) \Rightarrow k'(1)=?$

$k'(x)=f'(g(2x))\cdot g'(2x)\cdot(2)$

$\boxed{k'(1)}=f'(\underbrace{g(2)}_{1})\cdot g'(2)\cdot 2=2\,\underbrace{f'(1)}_{3}(5)=\boxed{30}$