Math 113 ; Dec. 1, 2020

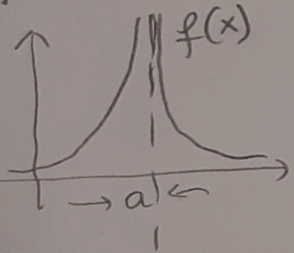
Infinite Limits and Limits at Infinity:

We will examine limits where the denominator approaches 0, but the numerator approaches a number different from zero as x becomes very large or very small, without bound. $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{M}{0} \right)$

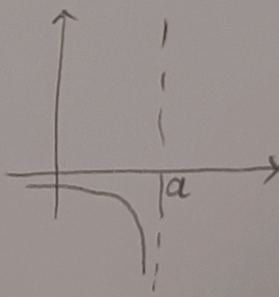
---

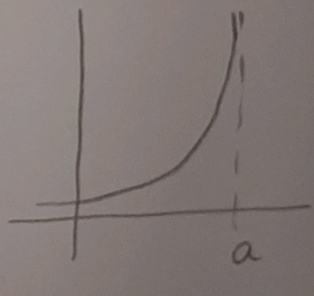
* If f increases (or decreases) without bounds as $x \to a$, we say that the limit is infinity (or -infinity) and write: $\lim_{x \to a} f(x) = \infty$ (or $\lim_{x \to a} f(x) = -\infty$).

* ∞ (or $-\infty$) is not a number. If limit is ∞ (or $-\infty$) it means limit does not exist D.N.E.



$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \infty$

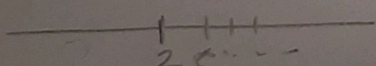




Examples: Evaluate the following limits:

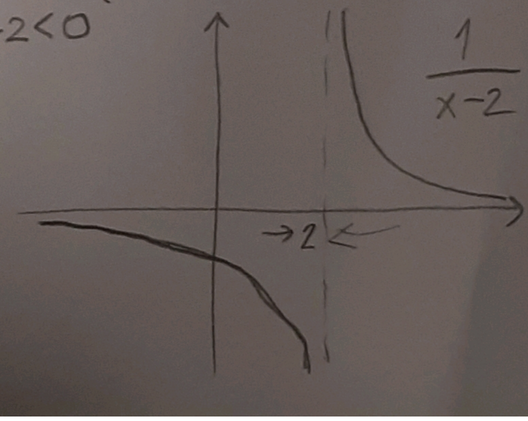
1) $\lim_{x \to 2^+} \left(\frac{1}{x-2} \right) = \frac{1}{0^+} = \infty$

$x - 2 > 0$



$\lim_{x \to 2^-} \left(\frac{1}{x-2} \right) = \frac{1}{0^-} = -\infty$

$x - 2 < 0$



(2) $\lim\limits_{x \to 1} \frac{1}{x-2} = D.N.E.$

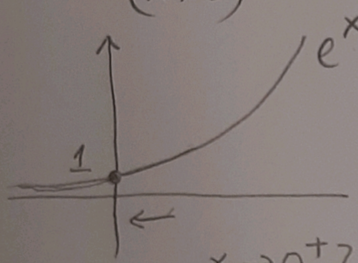
(3) $\lim\limits_{x \to 2} \frac{1}{(x-2)^2} = ?$

$\lim\limits_{\substack{x \to 2^- \\ x-2<0}} \frac{1}{(x-2)^2} = \frac{1}{(0^-)^2} = \infty$ | $\lim\limits_{\substack{x \to 2^+ \\ x-2>0}} \frac{1}{(x-2)^2} = \frac{1}{(0^+)^2} = \infty$

$\Longrightarrow$ $\boxed{\text{limit D.N.E.}}$ $\Rightarrow \lim\limits_{x \to 2} \frac{1}{(x-2)^2} = D.N.E.$

(4) $\lim\limits_{x \to 0^+} \left(\frac{e^x}{1-e^x}\right) = \frac{1^+}{0^-} = \boxed{-\infty}$

$(x>0)$



$\left.\begin{matrix} x \to 0^+ \\ x>0 \end{matrix}\right\} \; 1-e^x<0$

$\lim\limits_{\substack{x \to 0^- \\ x<0}} \left(\frac{e^x}{1-e^x}\right) = \frac{1}{0^+} = \boxed{\infty}$

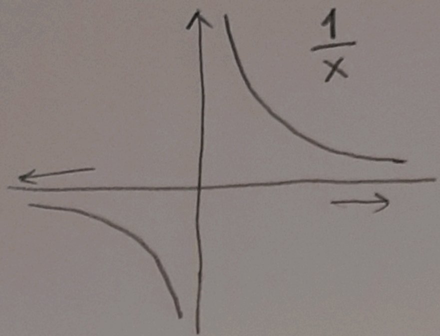
$e^x \to 1^- \Rightarrow 1-e^x = 0^+$

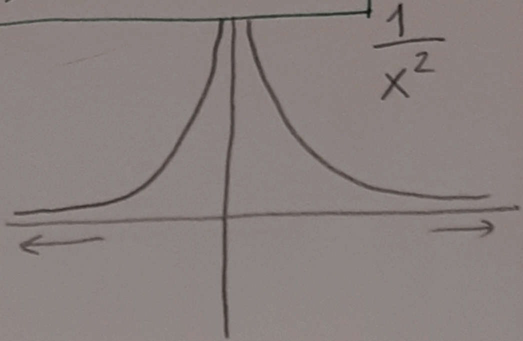
$\lim\limits_{x \to 0} \left(\frac{e^x}{1-e^x}\right) = \underline{\underline{d.n.e.}}$

Infinite Limits

Limits at Infinity: $(p>0)$

① $\boxed{\lim_{x\to\infty} \frac{1}{x^p} = 0}$ and $\boxed{\lim_{x\to -\infty} \frac{1}{x^p} = 0}$





---

② If $\deg(P(x)) > \deg(Q(x)) \Rightarrow \lim_{x\to \mp\infty} \left(\frac{P(x)}{Q(x)}\right) = \mp\infty$

③ If $\deg(P(x)) < \deg(Q(x)) \Rightarrow \lim_{x\to \mp\infty} \left(\frac{P(x)}{Q(x)}\right) = 0$

④ If $\deg P(x) = \deg Q(x) \Rightarrow \lim_{x\to \mp\infty} \left(\frac{P(x)}{Q(x)}\right) =$ ratio of leading term coefficients

---

Find the following limits:

① $\underbrace{\lim_{x\to\infty} \frac{1+x-2x^2}{3x^2+2x-5}}_{=} = \boxed{-\frac{2}{3}}$

$$\lim_{x\to\infty} \frac{\frac{1+x-2x^2}{x^2}}{\frac{3x^2+2x-5}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-\frac{1}{x}-2}{3+\frac{2}{x}-\frac{5}{x^2}} = \frac{0-0-2}{3+0-0}$$

$$= \boxed{\frac{-2}{3}}$$

(2) $$\lim_{x\to\infty}\left(\frac{2+\sqrt{x}+x^5}{\sqrt[3]{x}+1+2x^3}\right)=\infty \quad (\text{since } \deg P(x)=5>\deg Q(x)=3)$$

(3) $$\lim_{x\to\infty}\left(\frac{x^3+2x+\sqrt{x}}{2x^4+3x+5}\right)=0 \quad (\text{since } \deg P(x)=3<\deg Q(x)=4)$$

(4) $$\lim_{x\to-\infty}\left(\frac{x+9x^2}{3x^2+1}\right)=\frac{9}{3}=3$$

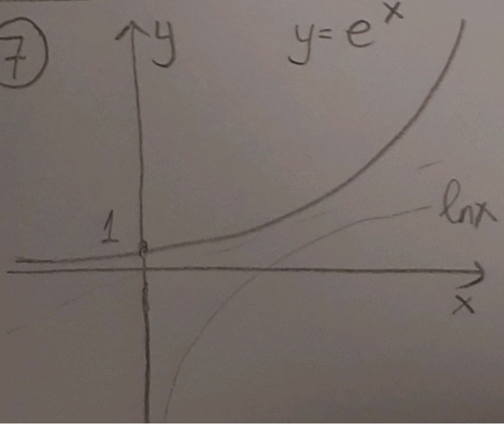
(5) $$\lim_{x\to-\infty}\left(\frac{1+x+3x^3}{x+x^2}\right)=-\infty \quad [\deg P(x)=3>\deg Q(x)=2]$$

$$=\lim_{x\to-\infty}\left(\frac{\frac{1}{x^3}+\frac{x}{x^3}+\frac{3x^3}{x^3}}{\frac{x}{x^3}+\frac{x^2}{x^3}}\right)=\lim_{x\to-\infty}\left(\frac{\overbrace{\cancel{\frac{1}{x^3}}}^{0}+\overbrace{\cancel{\frac{1}{x^2}}}^{0}+3}{\underbrace{\cancel{\frac{1}{x^2}}}_{0}+\underbrace{\cancel{\frac{1}{x}}}_{0}}\right)$$

$$=\frac{3}{\overset{+}{0}+0^-}=\frac{3}{0^-}=\boxed{-\infty}$$

(6) $$\lim_{x\to-\infty}\left(\frac{1+x+x^2}{x+x^3}\right)=\boxed{0} \quad (\deg P(x)=2<\deg Q(x)=3)$$

(7)



$$\lim_{x\to\infty} e^x=\infty$$

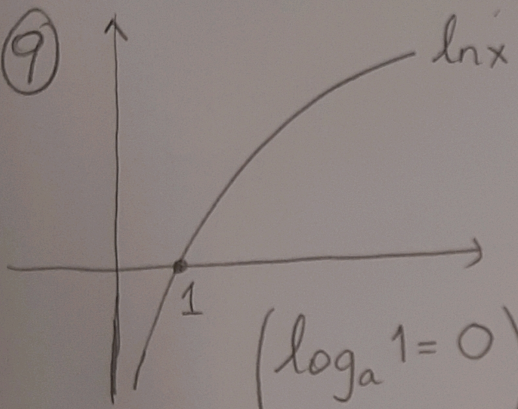
$$\lim_{x\to-\infty} e^x=\left(e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=\right)\boxed{0}$$

8. $\lim_{x\to\infty}\left(e^{-x}+3\right) = e^{-\infty}+3 = \frac{1}{e^{\infty}}+3 = 0+3 = 3$

$\lim_{x\to-\infty}\left(e^{-x}+2x+5\right) = e^{-(-\infty)}+2(-\infty)+5$

$= \underbrace{e^{\infty}}_{\downarrow\ \infty} - \infty + 5 = \boxed{\infty}$ (since $e^{\infty} \to \infty$ much faster than $2x \xrightarrow[x\to-\infty]{} -\infty$)

9.



$\left(\log_a 1 = 0 \quad a \neq 1, a > 0\right)$

$\lim_{x\to\infty}(\ln x) = \infty$

$\lim_{x\to 0^+}(\ln x) = -\infty$

---

10. $\lim_{x\to\infty}\left(-e^{-x^3}+9\right) = -e^{-\infty}+9 = -\frac{1}{e^{\infty}}+9 = 0+9 = 9$

$\lim_{x\to-\infty}\left(-e^{-x^3}+9\right) = -e^{-(-\infty)^3}+9 = -\underbrace{e^{+\infty}}_{\to\ \infty}+9$

$= -\infty+9 = \boxed{-\infty}$

Continuity:

A function $f(x)$ is continuous at $x=a$ if;

① f is defined at $x=a$

② $\lim_{x\to a} f(x)$ exists

③ $\lim_{x\to a} f(x) = f(a)$

If f is not cont. at $x=a \Rightarrow f$ is discontinuous at $x=a$.

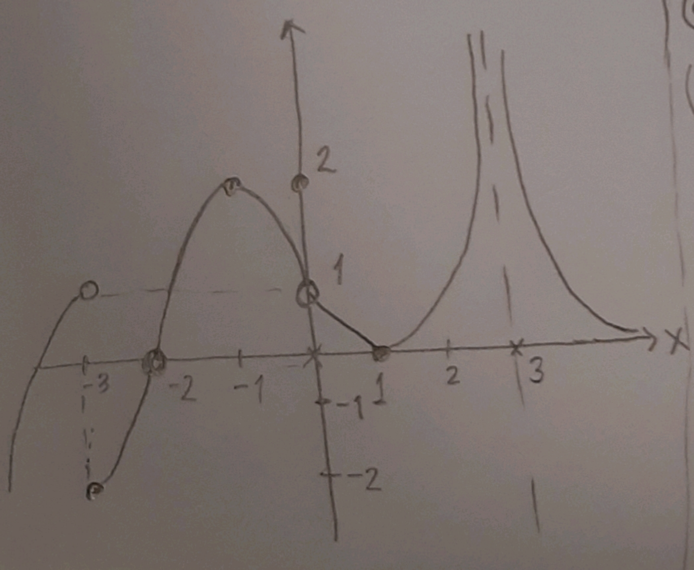
Note: * a polynomial function is cont. on $\mathbb{R}$

* $f(x)=e^x$ 〃 〃 〃 〃

* $f(x)=\ln(x)$ is cont. on $(0,\infty)$

* $f(x)=\dfrac{ax+b}{cx+d}$ is discont. when $x=-\dfrac{d}{c}$ (when denom. is zero)

① Find the points of discontinuity using the graph:



ⓐ $x=-3$: (i) $f(-3)=-2$

(ii) $\lim_{x\to -3^-} f(x)=1$, $\lim_{x\to -3^+} f(x)=-2$

$\Rightarrow \lim_{x\to -3} f(x) = $ d.n.e.

$\Rightarrow \boxed{f \text{ is discont. at } x=-3}$

b) $\underline{x=-2}$: $f(-2)$ is undefined $\Rightarrow$ $\boxed{f \text{ is discont. at } x=-2}$

---

c) $\underline{x=0}$: (i) $f(0)=2$ ✓

(ii) $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x)=1$, $\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x)=1 \Rightarrow \lim\limits_{x\to 0} f(x)=1$ ✓

(iii) $f(0)=2\neq 1=\lim\limits_{x\to 0} f(x)$

$\Rightarrow$ f is discont. at $x=0$

---

d) $\underline{x=3}$: (i) $f(3)$ is undefined

$\Rightarrow$ f is discont. at $x=3$

At all other points (i.e $\mathbb{R}\setminus\{-3,-2,0,3\}$) the function is continuous.

---

2) Find all discontinuities of $f(x)=\begin{cases} x^2+1 & \text{if } x\le 0 \\ 3x-1 & \text{if } 0<x<1 \\ \ln(e^{x+1}) & \text{if } x\geq 1 \end{cases}$

* x^2+1 and $3x-1$ are cont. on $\mathbb{R}$
* $\ln(e^{x+1})$ is cont. on $\mathbb{R}$

(since e^{x+1} is cont. on $\mathbb{R}$ and $>0 \Rightarrow e^{x+1}$ is in the domain of $\ln(u)$)

So, it is enough to check the jump points $x=0$ and $x=1$ in the defn. of $f(x)$

a) $x=0$:

i) $f(0)=0^2+1=1$ ✓

ii) $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x)=\lim\limits_{x\to 0^-}(x^2+1)=1$

$\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x)=\lim\limits_{x\to 0^+}(3x-1)=-1$

$\not\Rightarrow \lim\limits_{x\to 0} f(x)=$ d.n.e

$\Rightarrow$ $\boxed{f \text{ is discontinuous at } x=0}$

$(\ln e^{x+1}=x+1)$

b) $x=1$:

i) $f(1)=\ln e^{(1+1)}=\ln e^2=2$

ii) $\lim\limits_{\substack{x\to 1^- \\ (x<1)}} f(x)=\lim\limits_{x\to 1^-}(3x-1)=3(1)-1=2$

$\lim\limits_{\substack{x\to 1^+ \\ (x>1)}} f(x)=\lim\limits_{x\to 1^+}\ln e^{(x+1)}=2$

$\Rightarrow \boxed{\lim\limits_{x\to 1} f(x)=2}$

iii) $f(1)=2=\lim\limits_{x\to 1} f(x)$

$\Rightarrow$ f is continuous at $x=2$

So, the only pt. where f is discont. is $x=0$

$\boxed{f \text{ is cont. on } \mathbb{R}\setminus\{0\}}$

3) Find all discontinuities of $f(x)$ where

$$f(x) = \begin{cases} -x+1 & \text{if } x<1 \\ 5 & \text{if } x=1 \\ x^3-1 & \text{if } 1<x<4 \\ 63 & \text{if } x=4 \\ (x+3)(x+5) & \text{if } x>4 \end{cases}$$

f is cont. everywhere, except maybe at $x=1$ & $x=4$

at end pts:

$x=1$: $f(1)=5$, $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(-x+1) = -1+1=0$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^3-1) = 1-1=0$

$\lim_{x\to1} f(x) = 0$

$f(1)=5\neq 0=\lim_{x\to1} f(x) \Rightarrow$ f is discont. at $x=1$

$x=4$: $f(4)=63$

$\lim_{\substack{x\to4^- \\ (x<4)}} f(x) = \lim_{x\to4^-}(x^3-1) = 64-1=63$

$\lim_{\substack{x\to4^+ \\ (x>4)}} f(x) = \lim_{x\to4^+}[(x+3)(x+5)] = (4+3)(4+5) = 7\times9 = 63$

$\Rightarrow \lim_{x\to4} f(x) = 63 = f(4) \Rightarrow$ f is cont. at $x=4$

$\Rightarrow$ $\boxed{\text{the only discontinuity pt. of } f(x) \text{ is } x=1}$

4) Find the points of discont. of

$$f(x)=\begin{cases}\dfrac{|x|}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0.\end{cases} \qquad \left(\dfrac{|x|}{x}=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}\right)$$

i) $f(0)=1$

ii) $\lim\limits_{\substack{x\to 0^-\\(x<0)\\x\neq 0}} f(x)=\lim\limits_{x\to 0^-}(-1)=-1$

$\lim\limits_{\substack{x\to 0^+\\(x>0)\\x\neq 0}} f(x)=\lim\limits_{x\to 0^+}(1)=1$

$\lim\limits_{x\to 0} f(x)=$ d.n.e.

$\Rightarrow$ $\boxed{f \text{ is not cont. at } x=0}$

5) Find "a" and "b" such that $f(x)$ is cont. on $\mathbb{R}$:

$$f(x)=\begin{cases}ax^2 & \text{if } x\leq 3\\ \ln e^{(1+\frac{x}{3})} & \text{if } 3<x<6\\ \dfrac{b}{x} & \text{if } x\geq 6\end{cases}$$

I) $x=3$:

(i) $f(3)=a(3)^2=9a$

(ii) $\lim\limits_{x\to 3^-} f(x)=9a$

$\lim\limits_{\substack{x\to 3^+\\(x>3)}} f(x)=\lim\limits_{x\to 3^+}\ln e^{(1+\frac{x}{3})}=2$

$\lim\limits_{x\to 3} f(x)=\boxed{9a=2}$

$\Rightarrow a=\frac{2}{9}$

II) $\underline{x=6:}$

(i) $f(6)=\frac{b}{6}$

(ii) $\lim\limits_{\substack{x\to 6^- \\ (x<6)}} f(x)=\lim\limits_{x\to 6^-} \ln e^{(1+\frac{x}{3})}=3$

$\lim\limits_{\substack{x\to 6^+ \\ (x>6)}} f(x)=\frac{b}{6}$

$\lim\limits_{x\to 6} f(x)=\frac{b}{6}=3$

$\Downarrow$

$b=18$

$\Longrightarrow$ when $\boxed{a=\frac{2}{9} \text{ and } b=18}$ $f(x)$ is cont. on $\mathbb{R}$

---

6) Find the values of the constants "a" and "b" that make $f(x)$ cont. everywhere:

(Ans.: $a=\frac{1}{2}$, $b=\frac{1}{2}$)

$$f(x)=\begin{cases} \frac{x^2-4}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2\leq x\leq 3 \\ 2x-a+b & \text{if } x>3 \end{cases}$$

(I) $\underline{x=2:}$

(i) $f(2)=4a-2b+3$

(ii) $\lim\limits_{\substack{x\to 2^- \\ (x<2)}} f(x)=\lim\limits_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right)\underset{x\neq 2}{=}\lim\limits_{x\to 2^-}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}=2+2=4$

$\lim\limits_{\substack{x\to 2^+ \\ (x>2)}} f(x)=\lim\limits_{x\to 2^+}(ax^2-bx+3)=4a-2b+3$

to have a limit at $x=2 \Rightarrow$

$\boxed{4=4a-2b+3}$

$\Downarrow$

(*) $\boxed{4a-2b=1}$

(II) $x=3$:

(i) $f(3) = 9a - 3b + 3$

(ii) $\lim_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim_{x \to 3^+} (2x - a + b) = 6 - a + b$

to have a limit at $x=3$:

$9a - 3b + 3 = 6 - a + b$

$\Rightarrow \boxed{10a - 4b = 3}$ (**)

To find "a" and "b" we solve (*) & (**) together:

(*): $-2/\ 4a - 2b = 1 \Rightarrow -8a + 4b = -2$

(**): $10a - 4b = 3$ $\quad 10a - 4b = 3$

$2a = 1 \Rightarrow a = \frac{1}{2}$

$4a - 2b = 1 \Rightarrow 2b = 4a - 1 = 4\left(\frac{1}{2}\right) - 1 = 2 - 1 = 1$

$\Rightarrow 2b = 1 \Rightarrow b = \frac{1}{2}$

(7) Find "a" and "b" such that $f(x)$ is continuous everywhere:

$$f(x) = \begin{cases} 2 + ae^x & \text{if } x < 0 \\ b & \text{if } x = 0 \\ x^2 + 4 & \text{if } x > 0 \end{cases}$$

i) $f(0) = b$

ii) $\lim_{\substack{x \to 0^- \\ (x<0)}} f(x) = \lim_{x \to 0^-} (2 + ae^x) = 2 + a.e^0 = \boxed{2+a}$

$\lim_{\substack{x \to 0^+ \\ (x>0)}} f(x) = \lim_{x \to 0^+} (x^2 + 4) = \boxed{4}$

$2 + a = 4 \Rightarrow \boxed{a=2}$

$\Rightarrow \lim_{x \to 0} f(x) = 4 = 2 + a$

iii) $\lim_{x \to 0} f(x) = 4 = b = f(0)$

$\Rightarrow \boxed{b=4}$

8) Find the points of discontinuities of the following functions:

a) $f(x)=\frac{x}{x^2-1}$: $x^2-1=0 \Rightarrow x=\mp 1$

$\therefore f$ is not cont. $\boxed{\text{at } x=-1 \text{ and at } x=1}$

b) $f(x)=10^{\left(\frac{1}{x-3}\right)}$; $x-3=0 \Rightarrow x=3 \Rightarrow f$ is discont. $\boxed{\text{at } x=3}$

$\left(10^{\frac{1}{0}}=10^{\infty}=\infty\right)$

c) $f(x)=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{(x-2)}\underset{\text{if } x\neq 2}{\underset{\downarrow}{=}}(x+2)$ $\Big\}$ f is discont. $\boxed{\text{at } x=2}$

d) $h(x)=\frac{|x-2|}{x-2}$: $x-2=0 \Rightarrow x=2 \Rightarrow f$ is discont. $\boxed{\text{at } x=2}$

$$=\begin{cases} 1, & x>2 \\ -1, & x<2 \end{cases}$$

e) $f(x)=\frac{x^2+6x+9}{\underbrace{x^2+2x-15}_{(x+5)(x-3)}}$: $\Rightarrow f$ is discont. $\boxed{\text{at } x=3 \;\&\; x=-5}$

9) Is there any point at which $f(x)=\begin{cases} 1/x^2 & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$ is discont.?

$\boxed{\frac{1}{x^2} \text{ is cont. when } x\neq 0}$ $\Rightarrow$ So check $x=0$:

i) $f(0)=1$

ii) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\left(\frac{1}{x^2}\right)=\infty$, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\left(\frac{1}{x^2}\right)=\infty$

$\Rightarrow$ hence; $\lim\limits_{x\to 0} f(x)=\infty \Rightarrow$ i.e. limit D.N.E. at $x=0$

$\Rightarrow$ $\boxed{f \text{ is discont. at } x=0}$