Continuity:

f is cont. at $x=a$ iff $\boxed{\lim_{x\to a} f(x) = f(a)}$

That is;

* f is defined at $x=a$
* $\lim_{x\to a} f(x)$ should exist:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

* $\lim_{x\to a} f(x) = f(a)$

Example 5.7: HW

Ex. 5.8:

$$f(x) = \begin{cases} 2x^2+a, & \text{if } x<2 \\ b, & \text{if } x=2 \\ 3x-2, & \text{if } x>2 \end{cases}$$

Find "a" and "b" so that f is cont. at $x=2$

- $f(2) = b$
- $\lim_{x\to 2} f(x)$:

$$\lim_{\substack{x\to 2^- \\ (x<2)}} f(x) = \lim_{x\to 2^-} (2x^2+a) = \boxed{8+a}$$

$$\lim_{\substack{x\to 2^+ \\ (x>2)}} f(x) = \lim_{\substack{x\to 2^+ \\ (x>2)}} (3x-2) = \boxed{4}$$

For $f(x)$ to have a limit at $x=2$, we should have:

$$8+a=4 \Rightarrow \boxed{a=-4}$$

$$\Rightarrow \lim_{x\to 2} f(x) = \boxed{4}$$

- For continuity at $x=2$ we should have:

$$\underbrace{\lim_{x\to 2} f(x)}_{4} = \underbrace{f(2)}_{b}$$

$$\Rightarrow \boxed{b=4}$$

* At points where f is not continuous, it is said to be discontinuous.

Ex. 5.9: Let $f(x) = 2 + 12x - x^3 + 20x^4$.

Find the pts. where $f(x)$ is discontinuous.

Soln.: Since f is a polynomial, it is continuous everywhere.

Ex. 5.10: $f(x) = \dfrac{3x-2}{x^2+4}$.

f is discont. at pts. where denominator is zero.

But $x^2 + 4 \neq 0$ for any real number x.

So, $f(x)$ is cont. everywhere.

Ex. 5.11: $f(x) = \dfrac{x-2}{x^2-7x+10} = \dfrac{x-2}{(x-2)(x-5)}$

at $x=5$; function is not defined,

so f is discont. at $x=5$

($f(5)$ = undefined)

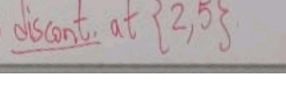
at $x=2$; $\lim\limits_{\substack{x \to 2 \\ x \neq 2}} \left[\dfrac{\cancel{(x-2)}}{\cancel{(x-2)}(x-5)}\right] = \dfrac{1}{2-5} = \dfrac{-1}{3}$

but $f(2)$ is undefined

so f is discont. also at $x=2$

$\Rightarrow$ f is cont. at $\mathbb{R} \setminus \{2, 5\}$

or at: $(-\infty, 2) \cup (2, 5) \cup (5, \infty)$



f is discont. at $\{2, 5\}$

Ex. 5.12: $f(x) = \begin{cases} \log\left(\frac{x}{2} + b\right), & x < 8 \\ x\left(\sqrt{x-8} + \frac{1}{4}\right), & x \geq 8 \end{cases}$

Find "b" if $f(x)$ is continuous at $x=8$.

(*) $f(8) = 8\left(\sqrt{8-8} + \frac{1}{4}\right) = \boxed{2}$

(*) $\lim\limits_{x \to 8} f(x) \Leftrightarrow \log_{10}(4+b) = 2 \Rightarrow 4 + b = 10^2 = 100$ $\boxed{b = 96}$

i) $\lim\limits_{\substack{x \to 8^- \\ (x<8)}} f(x) = \lim\limits_{\substack{x \to 8^- \\ (x<8)}} \left[\log_{10}\left(\frac{x}{2} + b\right)\right] = \boxed{\log_{10}(4+b)}$

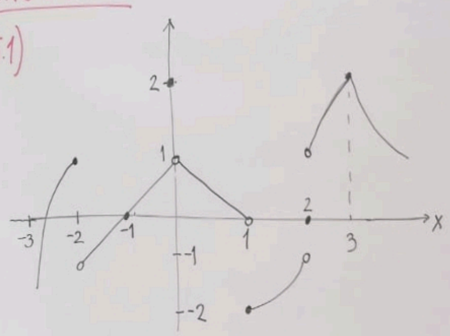
ii) $\lim\limits_{\substack{x \to 8^+ \\ (x>8)}} f(x) = \lim\limits_{\substack{x \to 8^+ \\ (x>8)}} \left(x\sqrt{x-8} + \frac{1}{4}\right) = \boxed{2}$

$\Rightarrow b = 96$ gives $\lim\limits_{x \to 8} f(x) = 2 = f(8)$

$\Rightarrow$ f is continuous at $x=8$ when $b=96$

Exercises:

5.1)



a) $\lim\limits_{\substack{x\to -2^- \\ (x<-2)}} f(x) = 1$, $\lim\limits_{\substack{x\to -2^+ \\ (x>-2)}} f(x) = -1 \Rightarrow \lim\limits_{x\to -2} f(x) = \text{d.n.e.}$

b) $\lim\limits_{\substack{x\to -1^- \\ (x<-1)}} f(x) = 0$, $\lim\limits_{\substack{x\to -1^+ \\ (x>-1)}} f(x) = 0 \Rightarrow \lim\limits_{x\to -1} f(x) = 0$

c) $\lim\limits_{\substack{x\to 0^- \\ (x<0)}} f(x) = 1$, $\lim\limits_{\substack{x\to 0^+ \\ (x>0)}} f(x) = 1 \Rightarrow \lim\limits_{x\to 0} f(x) = 1$

d.) $\lim\limits_{\substack{x\to 1^- \\ (x<1)}} f(x) = 0$ $\lim\limits_{\substack{x\to 1^+ \\ (x>1)}} f(x) = -2 \Rightarrow \lim\limits_{x\to 1} f(x) = \text{d.n.e.}$

e.) $\lim\limits_{\substack{x\to 2^- \\ (x<2)}} f(x) = -1$ $\lim\limits_{\substack{x\to 2^+ \\ (x>2)}} f(x) = 1 \Rightarrow \lim\limits_{x\to 2} f(x) = \text{d.n.e.}$

f.) $\lim\limits_{\substack{x\to 3^- \\ (x<3)}} f(x) = 2$ $\lim\limits_{\substack{x\to 3^+ \\ (x>3)}} f(x) = 2 \Rightarrow \lim\limits_{x\to 3} f(x) = 2$

5.2) At which pts. is $f(x)$ in Ex. 5.1 discont.? Explain why?

at $x=-2$: since $\lim\limits_{x\to -2} f(x) = \text{d.n.e.}$

at $x=0$: $2 = f(0) \neq \lim\limits_{x\to 0} f(x) = 1$

at $x=1$: $\lim\limits_{x\to 1} f(x) = \text{d.n.e.}$

at $x=2$: $\lim\limits_{x\to 2} f(x) = \text{d.n.e.}$

5.18) Find all discontinuities of $f(x)$:

$$f(x)=\frac{1}{1-|x|}$$

$x>0 \Rightarrow |x|=x \Rightarrow f(x)=\frac{1}{1-x} \Rightarrow$ discont. at $x=1$ $(x>0)$ ($f(1)$=undefined)

$x<0 \Rightarrow |x|=-x \Rightarrow f(x)=\frac{1}{1-(-x)}=\frac{1}{1+x} \Rightarrow$ discont. at $x=-1$ $(x<0)$ ($f(-1)$ is undefined)

$\Rightarrow f$ is discont. at $x=\mp 1$.

$\Rightarrow f$ is cont. on $\mathbb{R}\setminus\{\mp 1\}$

Find the values of constant(s) that will make the following functions cont. everywhere:

5.21)

$$f(x)=\begin{cases} a+bx^2, & \text{if } x<0 \\ b, & \text{if } x=0 \\ 2+e^{-x}, & \text{if } x>0 \end{cases}$$

* $f(0)=b$
* $\lim_{x\to 0} f(x)$: $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(a+bx^2)=a$

  $\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(2+e^{-x})=2+1=3$

  $\Rightarrow \lim_{x\to 0} f(x)=a=3$

* for cont. at $x=0$: $b=f(0)=\lim_{x\to 0} f(x)=3$

  $\Rightarrow b=3$

* at all other pts. $(x\neq 0)$ f is cont.

Hence, when $\boxed{a=b=3}$ f is cont. everywhere

5.22)

$$f(x)=\begin{cases} cx^2-2, & \text{if } x\leq 2 \\ \frac{x}{c}, & \text{if } x>2 \end{cases}$$

* $f(2)=c(2)^2-2=\boxed{4c-2}$
* $\lim_{x\to 2} f(x)=?$ i) $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(cx^2-2)=\boxed{4c-2}$ $(x<2)$

  ii) $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left(\frac{x}{c}\right)=\boxed{\frac{2}{c}}$ $(x>2)$

$\Rightarrow \lim_{x\to 2} f(x)=\boxed{\frac{2}{c}=4c-2} \Rightarrow$

$2=4c^2-2c \Rightarrow 2(2c^2-c-1)=0$

$2(2c+1)(c-1)$

$\Rightarrow 2c+1=0 \Rightarrow \boxed{c=-\frac{1}{2}}$

$c-1=0 \Rightarrow \boxed{c=1}$

$\Rightarrow$ So when $\boxed{c=1 \text{ or } c=-\frac{1}{2}}$ f is cont. at $x=2$

When $x\neq 2$; f is cont. (already)

$\Rightarrow f$ is cont. everywhere when $\boxed{c=1 \text{ or } c=-\frac{1}{2}}$

5.23) $f(x) = \begin{cases} x^2 - c^2, & \text{if } x \le 1 \\ (x-c)^2, & \text{if } x > 1 \end{cases}$

$* f(1) = 1^2 - c^2 = 1 - c^2$

$* \lim_{x \to 1} f(x) = ?$

i) $\lim_{\substack{x \to 1^- \\ (x<1)}} f(x) = \lim_{x \to 1^-} (x^2 - c^2) = 1 - c^2$

ii) $\lim_{\substack{x \to 1^+ \\ (x>1)}} f(x) = \lim_{x \to 1^+} (x-c)^2 = (1-c)^2$

for $\lim_{x \to 1} f(x)$ to exist:

$\Rightarrow 1 - c^2 = (1-c)^2$

$1 - c^2 = 1 - 2c + c^2$

$\Rightarrow 2c^2 - 2c + 1 - 1 = 0$

$2c(c-1) = 0$

$\Rightarrow \boxed{c = 0 \text{ or } c = 1} \Rightarrow \lim_{x \to 1} f(x)$ exists.

$\underline{c=0}$: $f(1) = 1 = \lim_{x \to 1} f(x)$

$\underline{c=1}$: $f(1) = 0 = \lim_{x \to 1} f(x)$

*At all other pts. $(x \neq 1)$ f is cont. $\Rightarrow$

$\Rightarrow$ Hence f is cont. everywhere when $\boxed{c = 0 \text{ or } c = 1}$

5.24) $f(x) = \begin{cases} e^{ax}, & \text{if } x \le 0 \\ \ln(b + x^2), & \text{if } x > 0 \end{cases}$

$* f(0) = e^{a(0)} = e^0 = \boxed{1}$

$* \lim_{x \to 0} f(x)$:

i) $\lim_{\substack{x \to 0^- \\ (x<0)}} f(x) = \lim_{x \to 0^-} (e^{ax}) = \boxed{1}$

ii) $\lim_{\substack{x \to 0^+ \\ (x>0)}} f(x) = \lim_{x \to 0^+} \ln(b + x^2) = \boxed{\ln b}$

$\Rightarrow \lim_{x \to 0} f(x) = \boxed{\ln b = 1} \Rightarrow b = e^1 = e$

$\therefore$ when $b = e$, $f(x)$ is cont. at $x = 0$

f is cont. everywhere else $(x \neq 0)$

$\therefore$ f is cont. everywhere when $b = e$